One Cannot Hear the Shape of a Drum.

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It is the vibrations in the drumhead which produce the sound in a drum. Certain characteristic frequencies determine how it vibrates and therefore how it sounds. In 1966, mathematician Mark Kac asked: Can you hear the shape of a drum? That is, if you know all the sounds the drum can make, can you infer the shape of a drum?

The following discussion is taken in large part from the article *You Can't Hear the Shape of a Drum* by Carolyn Gordon and David Webb [2].

We begin by considering the one-dimensional analogue of Kac's question: a vibrating string of length L. Idealize this string by the interval [0,L] and represent the possible configuration of the vibrating string as a function f(x,t) defined for x (the position variable) in [0,L] and any non-negative number t (the time variable). Since the endpoints are fixed, f must satisfy the boundary conditions f(0,t) = f(L,t) = 0 for all values of t. It can be shown that such a vibrating string must also satisfy the wave equation

$$\partial^2 f/\partial t^2 = \partial^2 f/\partial x^2$$

where ∂ is the change in the given variable.

Seeking stationary solutions of the form f(x,t) = g(x)h(t), our boundary conditions become g(0) = g(L) = 0 and the wave equation becomes g(x)h″(t) = g″(x)h(t) or

$$h''(t)/h(t) = g''(x)/g(x)$$

Since the left hand side does not depend on x, the right hand side cannot depend on x. Similarly, since the right hand side does not depend on t, the left hand side cannot depend on t. Thus, both sides must be some constant. It can be shown that this constant is negative. Denote it by $-\lambda$, where λ is positive. Our problem reduces to solving two equations:

$$g''(x) = -\lambda g(x) \quad (1)$$

$$h''(t) = -\lambda h(t) \quad (2)$$

We find that $\sin(\sqrt{\lambda}x)$ and $\cos(\sqrt{\lambda}x)$ are solutions of the spatial equation (1). The general solution turns out to be

$$g(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$$

where A and B are constants.

Now, $g(0) = 0 \Rightarrow B = 0$ so $g(x) = A\sin(\sqrt{\lambda}x)$ and
$g(L) = 0 \Rightarrow \sqrt{\lambda} = n\pi$ so $\lambda = n^2\pi^2/L^2$

The general solution the temporal equation (2) is

$$h(t) = C\sin(\sqrt{\lambda}t) + D\cos(\sqrt{\lambda}t)$$

where the constants C and D are determined by the initial configuration and velocity of the string. From here, it follows that the frequency is given by $\sqrt{\lambda}/2\pi$, where λ is the same as in

(1). Thus h represents a periodic oscillation of frequency $\sqrt{\lambda}/2\pi = n/2L$.
Thus the frequencies at which the string can vibrate are 1/2L, 1/L, 3/2L, ...

From the above, it is clear that the shape (length) of a stretched string can be determined to be half of the lowest frequency produced. In particular, one can hear the shape of a string.

The simple example of a vibrating string exhibits some important features of more complex vibrating systems:

1) The wave equation is linear: the sum of two solutions is again a solution and a constant multiple of a solution is again a solution.
2) The solutions $g(x) = A\sin(\sqrt{\lambda}x)$ of (1) satisfy a reflection principle: any function g solving (1) can be locally extended past its boundary (so (1) still holds) as the negative of its mirror reflection through its boundary (see figure 1).
(Set $g(0-x) = -g(0+x)$ and $g(L+x) = -g(L-x)$ for small $x>0$).

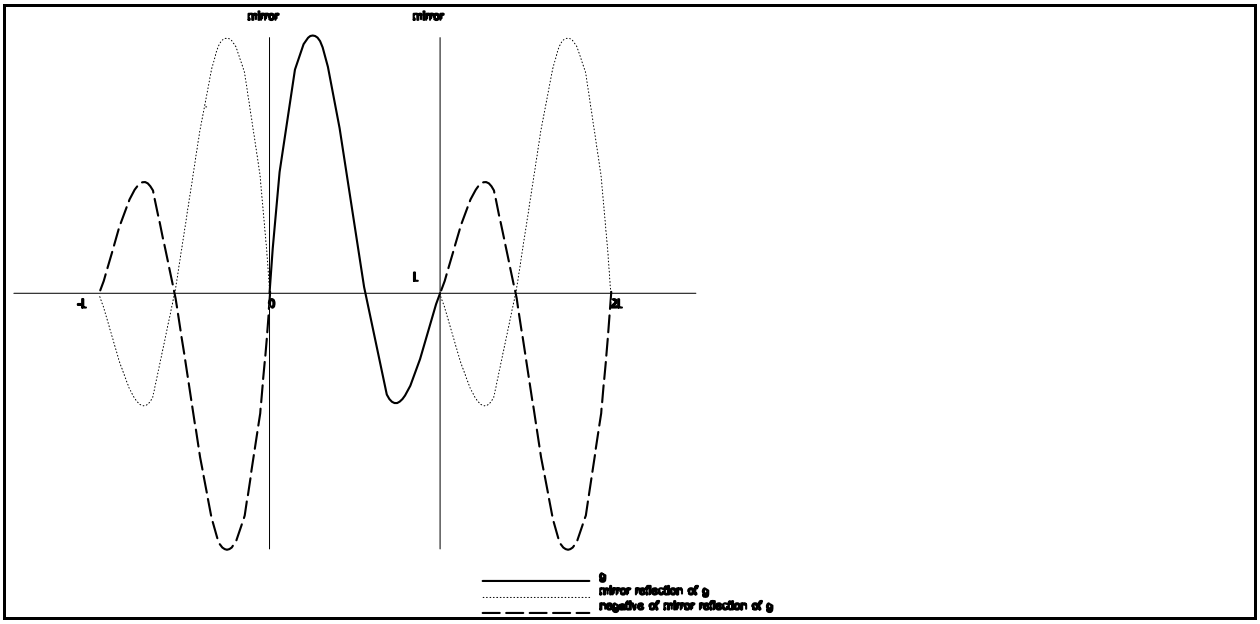


Figure 1

We now return to the original question. In his 1966 article, Kac was able to show that one can hear certain properties of a drum such as area and circumference [4]. Regarding shape, the answer came in 1991 in the work of Gordon, Webb, and Wolpert [3]. They found examples of differently shaped drumheads which have exactly the same characteristic vibration frequencies.

By the reflection principle, a waveform can be locally extended past a boundary edge as the negative of its mirror reflection through the boundary edge. Thus, for example, a waveform on the triangle L in figure 2 can be continued across the boundary edge to

admit an admissible waveform on the region made up of the two congruent triangles L and R. Simply specify the value of the function to be the negative of its value at the corresponding mirror point of L (see figure 3).

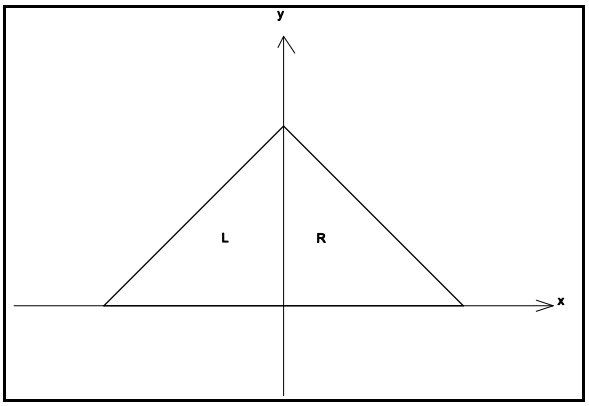


Figure 2

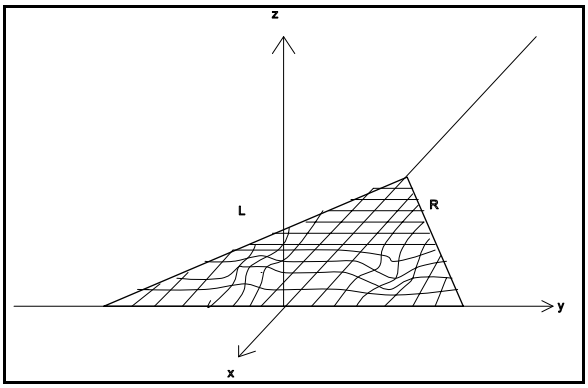


Figure 3

Imagine the right hand triangle, R, as the left hand triangle, L, flipped over and folded along the common edge of both triangles.

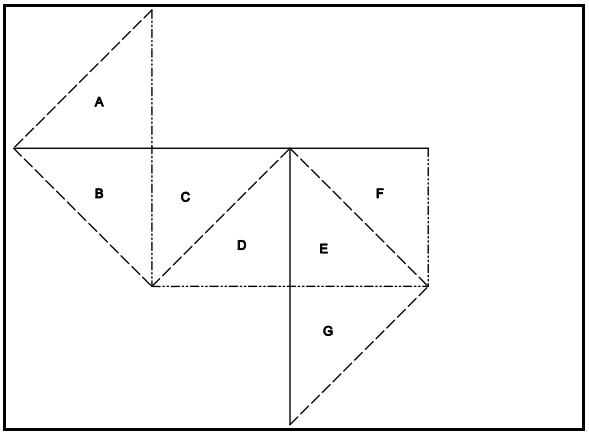


Figure 4

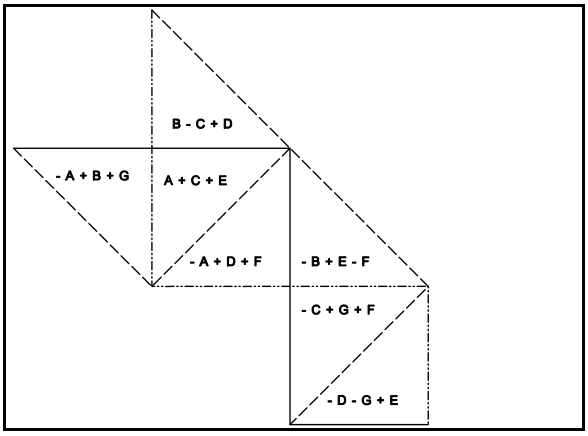


Figure 5

Consider the two regions D_1 and D_2 given above in figures 4 and 5 respectively. We will show that the drums defined by these regions (that is, drumheads fashioned in the shape of the regions) sound the same. That is, we will show that any waveform on the drum D_1 can be transplanted to a waveform with the same frequency on D_2 and vice versa so that the drums vibrate at precisely the same frequencies.

Consider a waveform ρ of a given frequency on the region D_1 and consider the portion of the graph of ρ which lies just above the triangle labelled A. Denote this piece of

the graph of ρ by A. Similarly, denote by B, C, ..., G the parts of the function ρ defined on triangles B, C, ..., G respectively. Then the graph of ρ is just the graphs of A, B, C, ..., G glued together along the interfaces between the triangles. In particular, since the waveform is a smooth function, the values of the functions A and B must coincide on their common edge. Also, since the boundary of D_1 remains fixed during the vibration, each function A, B, C, ..., G is zero along its outside edges.

Consider the function ψ on D_2 described in figure 5 specified by indicating a function on each of the triangles forming D_2. Thus, for example, on the topmost triangle in D_2, ψ is the function B-C+D (where -C is the negative of the mirror reflection of the function C), a superposition of the functions B, -C, and D. Ignoring boundary conditions, we see that ψ is a valid waveform on each of the individual triangles forming D_2. To see that ψ is a valid waveform over all of D_2, we must check that:

1) the seven pieces of ψ fit together smoothly at the interfaces between triangles.
2) the function ψ is zero on the boundary of D_2.

We verify both of the assertions by inspecting the relations of the seven functions arising from the structure of D_1. For example, check that the function B-C+D on the topmost triangle of D_2 and the function A+C+E on the neighbouring triangle fit together smoothly along their common interface, the solid edge. Reference to D_1 shows that triangles A and B share a common solid edge as do triangles D and E of D_1. So the functions A and B must be identical on their solid edge, as must D and E. Thus A+E and B+D will agree on the solid interface between the two triangles of D_2 in question. Now, note that the function C is zero on the solid edge (since it is a boundary edge). Thus, to continue it smoothly across the solid edge of the triangle below the topmost in D_2, we must put the function -C on the topmost triangle in accordance with the reflection principle. We find that B-C+D on the topmost triangle and A+C+E on the triangle below the topmost in D_2 fit together smoothly. It is easy to check (in the same way) that all seven pieces of ψ fit together smoothly, verifying our first assertion.

The second assertion can be verified in a similar manner. For example, we can show that ψ is zero on the dotted edge of the topmost triangle in D_2. Reference to D_1 shows that triangles C and D share a common dotted edge so they must coincide on that edge. Thus -C+D is zero on the dotted edge of the topmost triangle of D_2. Similarly, the dotted edge of triangle B in D_1 is a boundary edge from which it follows that B is zero on the dotted edge of the topmost triangle of D_2. Thus B-C+D is zero on the edge in question and so is ψ (by definition). In the same way it is easy to check that ψ is zero on all boundary edges of D_2.

We have shown that ψ is a valid waveform for the vibrating drumhead D_2. Also, the waveform ρ has been explicitly transplanted to a waveform ψ of the same frequency on D_2. Waveforms on D_2 can be similarly transplanted to D_1, so both drumheads vibrate at the same frequencies. In particular, we have found an example of two drumheads which sound the same; one cannot hear the shape of a drum.

Jon Chapman presents a method for finding a whole family of sound alike drums from D_1 and D_2 [1]. We can think of D_1 as being constructed from a single triangle T by a

series of reflections about its edges. Similarly for D_2. If we begin with a new triangle T^* and perform the same series of reflections on T^* as we did on T to form D_1, we obtain a new region D_1^*. Similarly, perform the same reflections as in the formation of D_2 to form D_2^*. We get drums D_1^* and D_2^* that sound the same. Thus we have found a whole family on sound-alike drums. In fact, the basic shape need not even be a triangle. We can use any shape with at least three edges. Simply pick three edges to represent the three sides of the triangle and reflect the shape about these edges following the same pattern of reflection that created D_1 and D_2 (see figures 6 and 7 - the cuts in the figures are of zero width and are included for clarity). Another set of sound alike drums results.

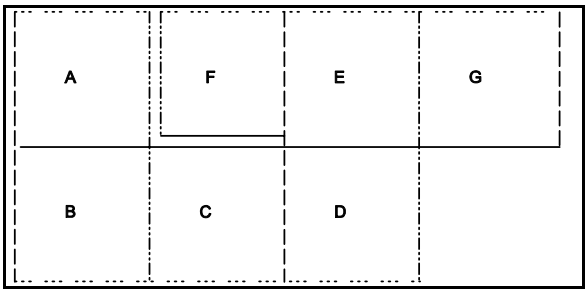


Figure 6

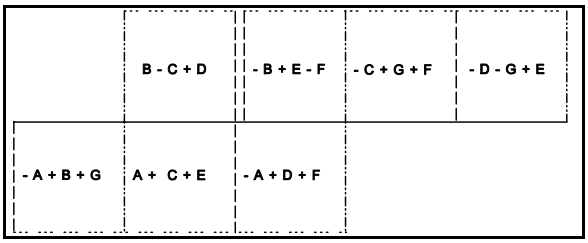


Figure 7

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1. Chapman, S. J., "Drums that Sound the Same", *American Mathematical Monthly*, 102, 124-138 (Feb 1995).

2. Gordon, C., Webb, D., "You Can't Hear the Shape of a Drum", *American Scientist*, 84, 1, 46-55 (Jan/Feb 1996).

3. Gordon, C., Webb, D., and Wolpert, S., "Isospectral Plane Domains and Surfaces via Riemannin Orbifolds", *Inventiones mathematicae*, 110, 1-22 (1992).

4. Kac, M., "Can One Hear the Shape of a Drum?", *American Mathematical Monthly*, 73, 4, 1-23 (1966).

5. www.ams.org/new-in-math/hap-drum/