Transit Time Distributions

Theory

A mathematical formulation of transit time distributions (TTDs), or in stratospheric terminology "age spectra", was presented by Hall and Plumb (1994), where the TTDs were identified as boundary Greens functions. This formulation is summarized below.
Consider the continuity equation for tracer concentration $\chi(r,t)$

$$\frac{\partial \chi}{\partial t} + L(\chi) = 0$$

where L is a linear transport operator. For tracers with concentration $\chi(\Omega,t)$ at the boundary Ω the response at an interior location r is

$$\chi(r,t) = \int_{-\infty}^{t} \chi(\Omega,\xi) G(r,t;\Omega,\xi) d\xi$$

where $G(r,t;\Omega,\xi)$ is the boundary Greens function, and satisfies the above continuity equation with boundary condition $G = \delta(t-\xi)$. G is known as the TTD or age spectrum. For stationary transport the above expression reduces to

$$\chi(r,t) = \int_{0}^{\infty} \chi(\Omega,t-\xi) G(r,\Omega,\xi) d\xi$$

also needs to be integrated over Ω

These expressions show that G 'propagates' mixing ratios on Ω at time ξ to location r at time t, i.e. G weights the contribution from Ω at various previous times to present time mixing ratio at r. Given the TTDs of a flow and the time history of a conserved passive tracer on Ω one can compute the tracer distribution throughout the flow using the above expressions.

Holzer and Hall (2000) have generalized the above to relate the transit time distribution to the consideration of explicit sources, rather than mixing ratio boundary conditions, while Haine and Hall (2002) have generalized the analysis to consider multiple source regions.

It is often useful to consider the temporal moments of the TTDs. By definition the zeroth moment of G is unity. The first moment is given by

$$\Gamma = \int_{0}^{\infty} G(\xi)\, \xi \, d\xi$$

is the mean transit time (or "mean age"). A measure of the spread of transit times is the second (centered) moment

$$2\Delta^2 = \int_{0}^{\infty} G(\xi)\, (\xi-\Gamma)^2 \, d\xi$$

where Δ is known as the width of the TTD.

One-dimensional Advection-Diffusion

For most realistic flows the TTDs cannot be determined analytically. However, analytic expressions are available for some idealized flows. One such flow is one dimensional flow with constant advective velocity U and diffusivity K. The tracer continuity for such a flow is

$$\frac{\partial \chi}{\partial t} + U\frac{\partial \chi}{\partial z} - K\frac{\partial^2 \chi}{\partial z^2} = 0$$

For this flow the TTD can be obtained using Laplace transforms (see Tim Hall's notes), and is

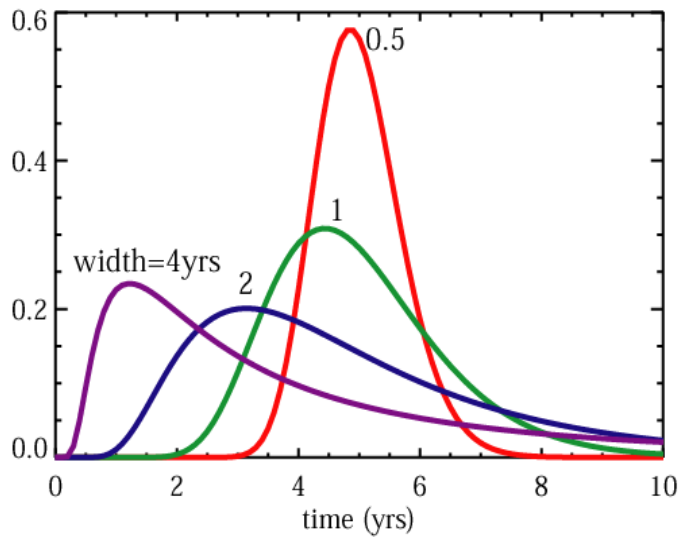
$$G(z,t) = \sqrt{\frac{z^2}{4\pi K t^3}}\ \exp\left(\frac{-(z-Ut)^2}{4Kt}\right)$$

which can be rewritten as

$$G(\Gamma,\Delta,t) = \sqrt{\frac{\Gamma^3}{4\pi \Delta^2 t^3}}\ \exp\left(\frac{-\Gamma(t-\Gamma)^2}{4\Delta^2 t}\right)$$

where the mean age $\Gamma = z/U$ and width $\Delta = 2Kz/U^3$. Distributions of the above form are known as Inverse Gaussian distributions, and have been used in many different fields to describe time distributions [e.g., Seshadri 1999].

For this flow the Peclet number $Pe = Uz/K = \Gamma^2/\Delta^2$. For fixed Γ, an increasing Δ corresponds to decreasing Peclet number and increasing role of diffusion relative to advection. The plot below shows TTDs of the above form for several values of width Δ with mean age Γ fixed. For small Δ transport is dominated by advection, and the TTD is narrow and peaked near the mean age ($\Delta = 0$ corresponds to no diffusion and a delta function TTD at $t=\Gamma$). As Δ is increased (diffusion increased) the TTD broadens, has a peak (``modal time'') at transit times increasingly shorter than the mean age, and develops an increasingly longer ``tail'' of old fluid.



TTDs for 1-D advection-diffusion model with mean age Γ = 5 years and width Δ = 0.5, 1, 2, and 4 years.

Tracer Ages

The TTD framework can be used to interpret timescales derived from chemical tracers with time varying sources or sinks (so called "transient tracers"). In the special case of advective flow G is a delta function peaked at advective time τ_{adv}. The mixing ratio at r is then simply $\chi(r,t) = \chi(\Omega,t-\tau_{adv})$, and all tracer signals propagate at the rate. However for flows with mixing G has finite width and different tracer signals propagate at different rates.

To illustrate this consider idealized tracers which are conserved and have exponential growth λ. For these tracers it is possible to a define a tracer age $\tau(r)$ as the elapsed time since the surface concentration was equal to the concentration at r, i.e., $\chi(r,t) = \chi(\Omega,t-\tau(r))$. Using the above expressions it can be shown that the tracer age is given by

$$\tau = -\lambda^{-1}\int_0^\infty G(\xi)\, e^{-\lambda\xi}\, d\xi$$

(this is also the tracer age for tracers with constant surface concentrations and radioactive decay λ).

Expanding G in terms of moments and neglecting yields moments higher than two yields

$$\tau \sim \Gamma - \lambda^{-1} \ln (1 + \lambda^2 \Delta^2) \sim \Gamma - \lambda\Delta^2$$

where $\lambda\Delta << 1$ is assumed for the second approximation.

Consider first the special limit of a tracer with linear growth. In this limit the tracer age τ equals the mean age Γ regardless of the shape of the TTDs (i.e., independent of Δ and higher moments). (This can be seen in the above equations by letting $\lambda \to 0$.) So conserved tracers with approximately linear growth over the width of the TTDs can be used to estimate the mean transit time.

For non-zero λ, the tracer age τ is less than the mean age Γ, for nonzero Δ. There is larger sensitivity to the shape of the TTDs (i.e., Δ) for more rapid growth/decay. The above approximations of τ can be used to explain differences in the tracer ages derived from different tracers (e.g., different CFCs), see transient tracers section.

Approximate expressions for tracer ages for other idealized tracers are derived in Waugh et al., JGR, 2003.

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