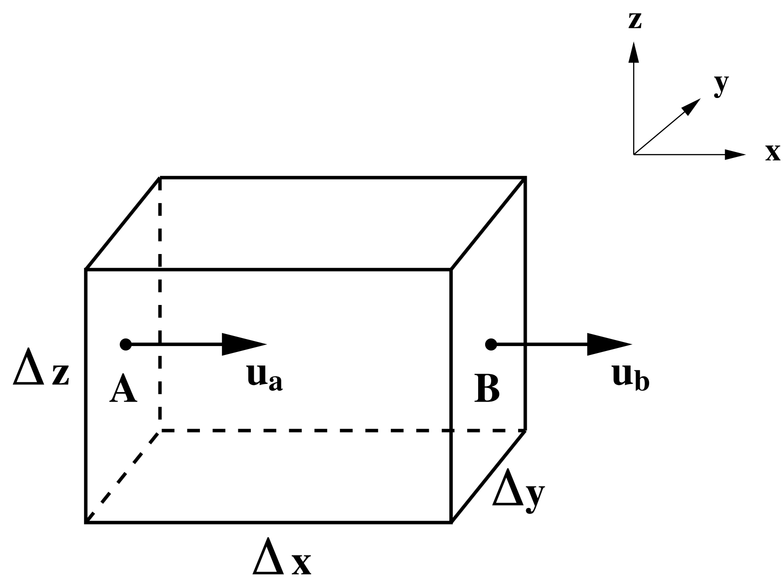
Derivation of the Continuity and Navier Stokes Equations

1 Continuity equation

Consider a small volume of fluid of volume V and sides Δx, Δy, Δz.



The mass flux (flow per unit time) out of box through side B is

$$\rho_B u_b \Delta y \Delta z$$

and through side A

$$-\rho_A u_A \Delta y \Delta z$$

Summing over all sides will give the **rate of decrease of mass per unit time**, i.e.

$$-\frac{\partial}{\partial t}\int_V \rho dV = [\Delta(u\rho)\Delta y\Delta z + \Delta(v\rho)\Delta z\Delta x + \Delta(w\rho)\Delta x\Delta y]$$

As V $\rightarrow$ 0

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(u\rho)}{\partial x} + \frac{\partial(v\rho)}{\partial y} + \frac{\partial(w\rho)}{\partial z}\right]$$

More generally

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{u})$$

(see crib sheet for definition of vector operators)

If ρ is constant then

$$\nabla \cdot \underline{u} = 0$$

The above holds (approximately) even when the density of the fluid varies, so long as pressure effects are negligible. The flow is then said to be **incompressible**.

The assumption of **incompressibility** usually holds provided the flow speed is much smaller than the speed of sound, which is around 340 m s^{-1} in air and 1470 m s^{-1} in water.

2 Momentum equation – Navier Stokes equation

Newton's second law of motion applied to a fluid particle reads

> The rate of change of momentum of a fluid particle is **equal** to the net force acting upon it

All we need do is determine both sides of the equality.

2.1 Material Derivative

Consider the rate of change of a scalar quantity T. A small change

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

The rate of change following the fluid motion is given by dividing by δt, making $\delta t \to 0$, and setting $\partial x/\partial t = u$, etc. Thus

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

which can be written

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + (\underline{u} \cdot \nabla)T$$

The operator D/Dt is known as the **material derivative** or the **rate of change following a fluid particle**. The change in a property of the fluid at a point is brought about by **local changes with time** and the **advection** of the property past the point.

If

$$\frac{DT}{Dt} = 0$$

then the property of the fluid particle is unchanged with time.

For the velocity of the fluid, by similar reasoning, we have

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}$$

The change in momentum *per unit volume* following the fluid motion is then

$$\rho\frac{D\underline{u}}{Dt} = \rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u} \cdot \nabla)\underline{u}$$

2.2 Forces acting on a fluid particle

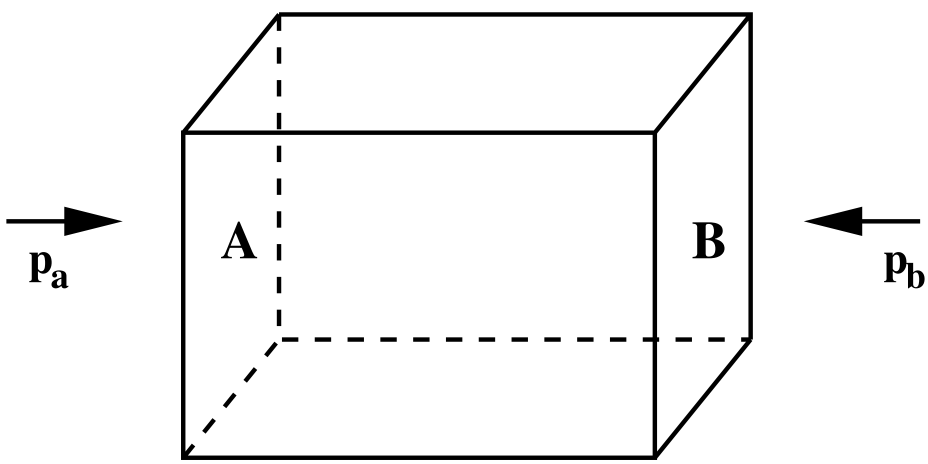
These can be divided into two kinds:

External – such as gravity

and

Internal – such as pressure and viscosity

2.2.1 Pressure



If $p_A = p_B$, where p_n is the pressure exerted on a given side of the fluid element, then there is no net force acting on the fluid element.

The net force acting in the x direction is

$$(p_A - p_B)\delta y\delta z$$

$$= -\frac{\partial p}{\partial x}\delta x\delta y\delta z$$

The net force in the x direction per unit volume is

$$-\frac{\partial p}{\partial x}$$

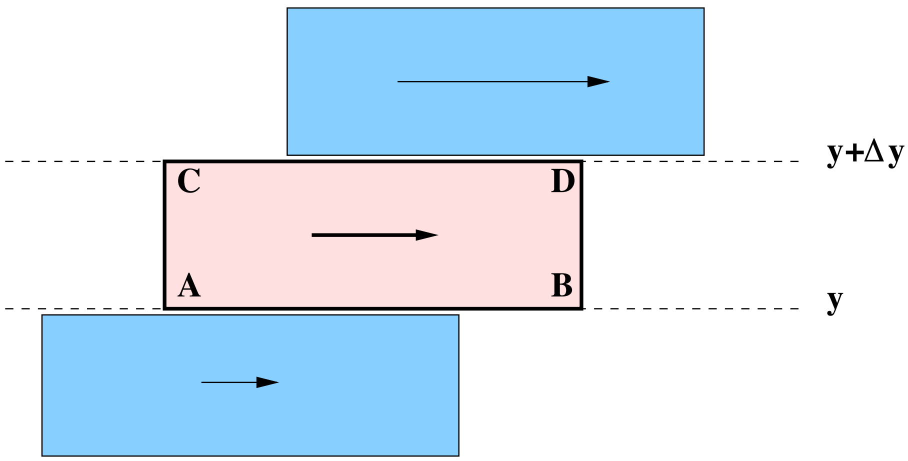
and the total force *per unit volume* is

$$-\nabla p$$

2.2.2 Viscosity

Viscous stresses oppose the relative movement between neighbouring fluid particles.

Consider a **plain straining motion**



The faster fluid above **AB** will drag forward the slower moving fluid below, and visa versa.

In a **Newtonian fluid** the stress is directly proportional to the velocity gradient (strain). Thus

$$\tau = \mu \frac{\partial u}{\partial y}$$

where μ is the **coefficient of viscosity** of the fluid and τ the force per unit area.

The force acting across **AB** is

$$\mu \left(\frac{\partial u}{\partial y} \right)_y \delta x \delta z$$

across **CD** is

$$\mu \left(\frac{\partial u}{\partial y} \right)_{y+\Delta y} \delta x \delta z$$

so that the **net** force *per unit volume* on our element is

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\delta x\delta y\delta z$$

The general case is mathematically more difficult. However to a good approximation, for most practical cases, the viscous force per unit volume is simply

$$\mu\nabla^2\underline{u}$$

Collecting all the terms together we get

$$\rho\frac{D\underline{u}}{Dt} = -\nabla p + \mu\nabla^2\underline{u} + \underline{F}$$

where $\underline{F}$ is referred to as a body force (example – gravity, $\underline{g}$). This equation is known as the **Navier Stokes equation**. Often the equation is divided through by ρ and $\nu = \mu/\rho$ is known as the **kinematic viscosity**.