Equations of motion

The waters of the oceans are continually in motion driven primarily by the atmosphere through winds blowing across the ocean surface and differential heating and cooling. In order to study the dynamics of these motions of the ocean and to understand how the ocean will respond to a given forcing we require a system of equations that describe the fluid motion. There are two problems facing us; one is to couch the equations in a form suitable for an observer on the rotating earth, the other is to restrict the dynamics of the system to those scales of motion that we are interested in. In this course we are concerned with scales of motion in the ocean from a few tens of kilometres to the size of an ocean basin (we will be more precise in the definition of these scales later).

1 Equations of motion for a fluid on a rotating sphere

First we consdier the momentum equation.

Momentum equation

The momentum equation is simply a statement of Newton's second law of motion, which in a fixed frame of reference is

$$\frac{d_f \underline{u}_f}{dt} = \sum_i \underline{F}_i$$

where $\underline{u}_f$ is the velocity of a particle (the subscript f referring to the fixed frame of reference) and $\underline{F}_i$ the forces per unit mass acting on that particle.

For a fluid this becomes the Navier Stokes equation

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u} + \frac{\underline{F}}{\rho}$$

where $\underline{u}$ is the fluid velocity, p the pressure, ρ the fluid density, ν the kinematic viscosity and $\underline{F}$ any external forces acting on the fluid such as gravity, $\rho \underline{g}$.

We want to work in a frame of reference relative to the rotating Earth. Let the angular velocity of the rotating frame be $\underline{\Omega}$.

Then a point, $\underline{x}_r$, with a fixed position in the rotating frame has a velocity $\underline{\Omega} \wedge \underline{x}_r$ (fig. 1). When the point $\underline{x}_r$ is moving relative to the rotating frame (i.e. moving over the surface of the Earth), the velocity relative to the fixed frame is

$$\frac{d\underline{x}_f}{dt} = \frac{d\underline{x}_r}{dt} + \underline{\Omega} \wedge \underline{x}_r \quad .$$

Repeating the operation gives the acceleration of the particle. Thus

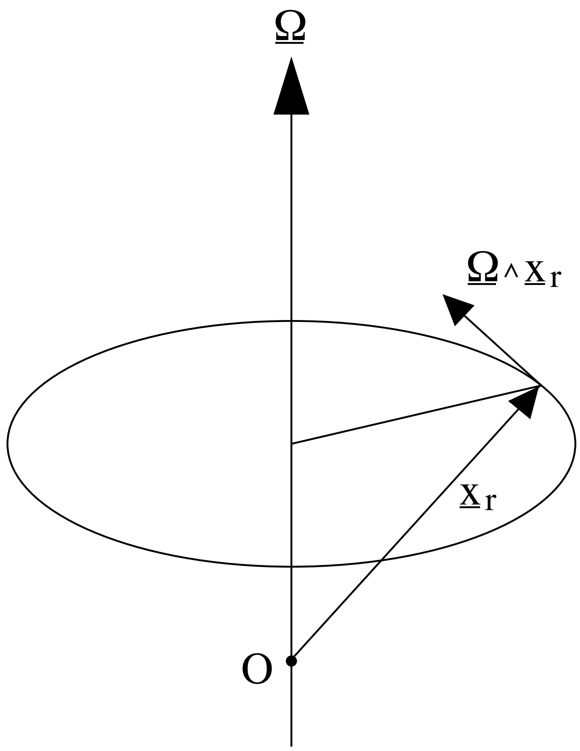

Figure 1: Rotating system

$$\begin{aligned}\frac{d^2\underline{x}_f}{dt^2} &= \frac{d}{dt}\left(\frac{d\underline{x}_r}{dt} + \underline{\Omega}\wedge\underline{x}_r\right) + \underline{\Omega}\wedge\left(\frac{d\underline{x}_r}{dt} + \underline{\Omega}\wedge\underline{x}_r\right)\\ &= \frac{d^2\underline{x}_r}{dt^2} + 2\underline{\Omega}\wedge\frac{d\underline{x}_r}{dt} + \underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{x}_r)\ .\end{aligned}$$

Using the vector identity $\underline{\Omega}\wedge(\underline{\Omega}\wedge\underline{x}_r) = \nabla(\Omega^2 x_r^2/2)$ and writing $d\underline{x}/dt = \underline{u}$, we get

$$\frac{d\underline{u}_f}{dt} = \frac{d\underline{u}_r}{dt} + 2\underline{\Omega}\wedge\underline{u} + \nabla\left(\frac{1}{2}\Omega^2 x_r^2\right)\ .$$

When the point $\underline{x}_r$ is the position of a material volume element of the fluid then the derivative d/dt is the same as the substantive derivative $D/Dt = \partial/\partial t + \underline{u}\cdot\nabla$, or the rate of change flowing the fluid motion.

The **momentum equation** for a fluid in a rotating frame of reference is then

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\wedge\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi_v\ . \quad (1)$$

The subscript r has been dropped and from now on will be implied in all that we do. Φ_v is referred to as the total gravity, and is the sum of the geopotential due to gravity and the rotational acceleration, i.e.

$$\Phi_v = \Phi - \frac{1}{2}\Omega^2 x_r^2$$

where $\nabla\Phi = \underline{g}$. Note that we have ignored any forcing and dissipative processes. These will be considered later.

We require additional equations to fully describe the system. These are similar to those in a non–rotating system and are:

Conservation of mass

or the continuity equation

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \underline{u} = 0 \tag{2}$$

The equation of state

relating density to fluid properties such as pressure p, temperature T and salinity S

$$\rho = \rho(p, S, T) \tag{3}$$

and

Conservation of state variables

such as temperature and salinity

$$\frac{DT}{Dt} = 0 \tag{4}$$

$$\frac{DS}{Dt} = 0 \ \ . \tag{5}$$

Again we have ignored for the moment any dissipative processes.

As in many branches of fluid dynamics we will make two basic assumptions to simplify the system a little.

Incompressibility

For many purposes the fact that sea water, and even air, is compressible is not relevant for the dynamics of many of the motions that occur in the ocean and atmosphere. The change in volume of a fluid particle undergoing modest vertical excursions is usually negligible. In this case the continuity equation reduces to a simple statement of the non-divergence of the velocity field,

$$\nabla \cdot \underline{u} = 0 \ \ . \tag{6}$$

The validity of the incompressibiltiy assumption formally requires that

1. particle velocities are small compared with the speed of sound, c_s

2. the phase speed of waves in the system $<< c_s$
3. the vertical scale of motion $<<$ the scale height H_s ($\simeq \rho/(d\rho/dz)$), the height over which the density varies appreciably

   ($H_s \simeq 200$ km for the ocean, 10 km for the atmosphere).

Boussinesq

In the ocean the density ρ never departs more than 2% from its mean value ρ_o. Under the Boussinesq approximation the density is taken to be constant in computing rates of change of momentum. In the momentum equation density variations are only taken into account when they give rise to buoyancy forces through the gravitional term. This approximation requires that the vertical scale of the vertical component of velocity, w, is small compared with H_s.

Equations (1-5) cover a vast range of motions of a fluid on the Earth from the planetary scale down to small turbulent eddies. Even with the above approximations the solution of the equations is generally intractable. In any case the richness of solutions will mask the particular process we may be interested in. We will simplify the equations by considering the expected magnitude of individual terms. But first we must choose a coordinate system. The natural choice for the Earth are spherical coordinates.

2 Equations of motion in spherical coordinates

The coordinates are taken as (r, θ, ϕ) representing the distance from centre of the Earth, latitude and longitude, respectively (fig. 2). The velocity components (u, v, w) in each coordinate direction then represent eastward, northward and vertical motions.

The equations of motion can then be written in the component form thus

Continuity equation

$$\frac{1}{r\cos\theta}\frac{\partial u}{\partial \phi} + \frac{1}{r\cos\theta}\frac{\partial (v\cos\theta)}{\partial \theta} + \frac{2w}{r} + \frac{\partial w}{\partial r} = 0 \qquad (7)$$

Momentum equations

$$\frac{Du}{Dt} + \frac{uw}{r} - \frac{uv}{r}\tan\theta - 2\Omega v\sin\theta + 2\Omega w\cos\theta = -\frac{1}{\rho_o r\cos\theta}\frac{\partial p}{\partial \phi} \qquad (8)$$

$$\frac{Dv}{Dt} + \frac{vw}{r} + \frac{u^2}{r}\tan\theta + 2\Omega u\sin\theta = -\frac{1}{\rho_o r}\frac{\partial p}{\partial \theta} \qquad (9)$$

$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\theta = -\frac{1}{\rho_o}\frac{\partial p}{\partial r} - g \qquad (10)$$

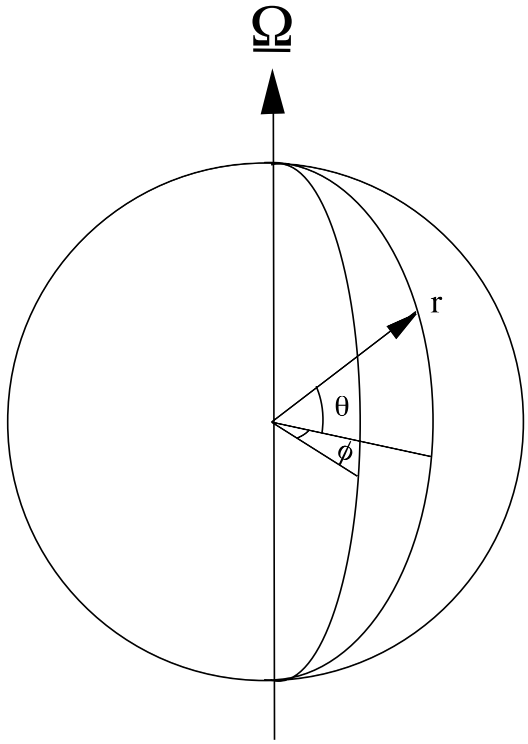

Figure 2: Spherical coordinate system

where the substantive derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial}{\partial\phi} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial r} \quad .$$

Because we will be referring to distances with respect to some reference latitude, θ_o, and the distance from the surface of the Earth rather than its centre we will introduce the new coordinates

$$\begin{aligned} x &= \phi a \cos\theta_o \\ y &= (\theta - \theta_o) a \\ z &= r - a \end{aligned}$$

where a is the mean radius of the Earth.

Derivatives in the ϕ, θ and r directions can then be written as

$$\begin{aligned} \frac{\partial}{\partial\phi} &= a\cos\theta_o\frac{\partial}{\partial x} \\ \frac{\partial}{\partial\theta} &= a\frac{\partial}{\partial y} \\ \frac{\partial}{\partial r} &= \frac{\partial}{\partial z} \end{aligned}$$

and the substantive derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\cos\theta_o}{\cos\theta}\frac{a}{r}\frac{\partial}{\partial x} + v\frac{a}{r}\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad .$$

The coordinates (x, y, z) are approximately a Cartesian system if $\theta - \theta_o << 1$ (i.e. small departures from the reference latitude) and $z/a << 1$ (the depth of the ocean $D \simeq 5$ km whilst the radius of the Earth $a \simeq 6370$ km).

With these restrictions

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{2w}{r} - v\frac{\tan\theta}{a} = 0 \tag{11}$$

and the horizontal components of the momentum equation

$$\frac{Du}{Dt} + \frac{uw}{a} - \frac{uv}{a}\tan\theta - 2\Omega v\sin\theta + 2\Omega w\cos\theta = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} \tag{12}$$

$$\frac{Dv}{Dt} + \frac{vw}{a} + \frac{u^2}{a}\tan\theta + 2\Omega u\sin\theta = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} \quad . \tag{13}$$

Scale analysis

To proceed we consider the magnitude of each term in the equations by specifying typical scales of length, time and velocity for the ocean at mid–latitude ($\theta \simeq 45^o$):

Horizontal velocity,	U	$\sim$	0.1 ms^{-1}
Vertical velocity,	W	$\sim$	10^{-4} ms^{-1}, (10 m/day)
Horizontal length,	L	$\sim$	10^5 m
Vertical length,	D	$\sim$	5 x 10^3 m
Time scale (advection),	L/U	$\sim$	10^6 s.

The radius of the Earth we take as $a = 6$ x 10^6 m and the rotation rate $\Omega = 2\pi/\text{period} = 7$ x 10^{-5} s^{-1}.

Then for the continuity equation

term	$\frac{\partial u}{\partial x}$	$\frac{\partial v}{\partial y}$	$\frac{\partial w}{\partial z}$	$\frac{2w}{a}$	$\frac{v\tan\theta}{a}$
scale	$\frac{U}{L}$	$\frac{U}{L}$	$\frac{W}{D}$	$\frac{2W}{a}$	$\frac{U}{a}$
magnitude	10^{-6}	10^{-6}	2 x 10^{-8}	3 x 10^{-11}	10^{-8}

Retaining the highest terms the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{14}$$

For the u component of the momentum equation

term	$\frac{Du}{Dt}$	$\frac{uw}{a}$	$\frac{uv}{a}\tan\theta$	$2\Omega v\sin\theta$	$2\Omega w\cos\theta$	$\frac{1}{\rho}\frac{\partial p}{\partial x}$
scale	$\frac{U^2}{L}$	$\frac{UW}{a}$	$\frac{U^2}{a}$	fU	fW	?
magnitude	10^{-7}	10^{-9}	10^{-9}	10^{-5}	10^{-8}	

A similar scaling applies to the v component o fthe momentum equation. The only term that can balance the highest rotation term, the Coriolis term, is the pressure gradient, i.e.

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{15}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{16}$$

where $f = 2\Omega\sin\theta$, the Coriolis parameter. This balance of terms is known as the **geostrophic relationship**. Velocities that satisfy this relationship are known as **geostrophic velocities**.

Although the geostrophic relationship is a strong balance it cannot tell us how the flow will evolve with time as it contains no time derivative. We therefore need to retain the next highest term to get

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{17}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad . \tag{18}$$

The ratio of the advection to the Coriolis term is

$$\frac{\text{advection}}{\text{Coriolis}} = \frac{U^2/L}{fU} = \frac{U}{fL}$$

is known as the **Rossby number**, R. For the choice of scales we made above $R \simeq 0.1$.

Lastly for the vertical component of the momentum equation

term	$\frac{Dw}{Dt}$	$\frac{u^2+v^2}{a}$	$2\Omega u\cos\theta$	$\frac{1}{\rho}\frac{\partial p}{\partial z}$	g
scale	$\frac{UW}{L}$	$\frac{U^2}{a}$	fU	?	g
magnitude	10^{-10}	10^{-9}	10^{-5}		10

The only term that can balance gravity is the vertical pressure gradient, i.e.

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \quad = \quad -g \quad . \tag{19}$$

Integrating from level z_o we find

$$p = p_o - \int_{z_o}^{z} \rho g dz$$

i.e. the pressure at any point is equal to the weight of water above it. This approximation is known as the **hydrostatic approximation**. We are ignoring any changes to pressure cuased by the movement of the fluid. The approximation is formally correct when the ratio of vertical to horizontal scales of motion tends to zero.

The set of equations we shall be using is given below. The approximations applied are equivalent to including only the locally vertical component of rotation and using a local Cartesian coordinate frame (i.e. ignoring the metric terms).

Horizontal momentum equations

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} \tag{20}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} \tag{21}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

and the Coriolis parameter $f = 2\Omega \sin\theta$.

Hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g \tag{22}$$

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{23}$$

Conservation of density

$$\frac{D\rho}{Dt} = 0 \tag{24}$$

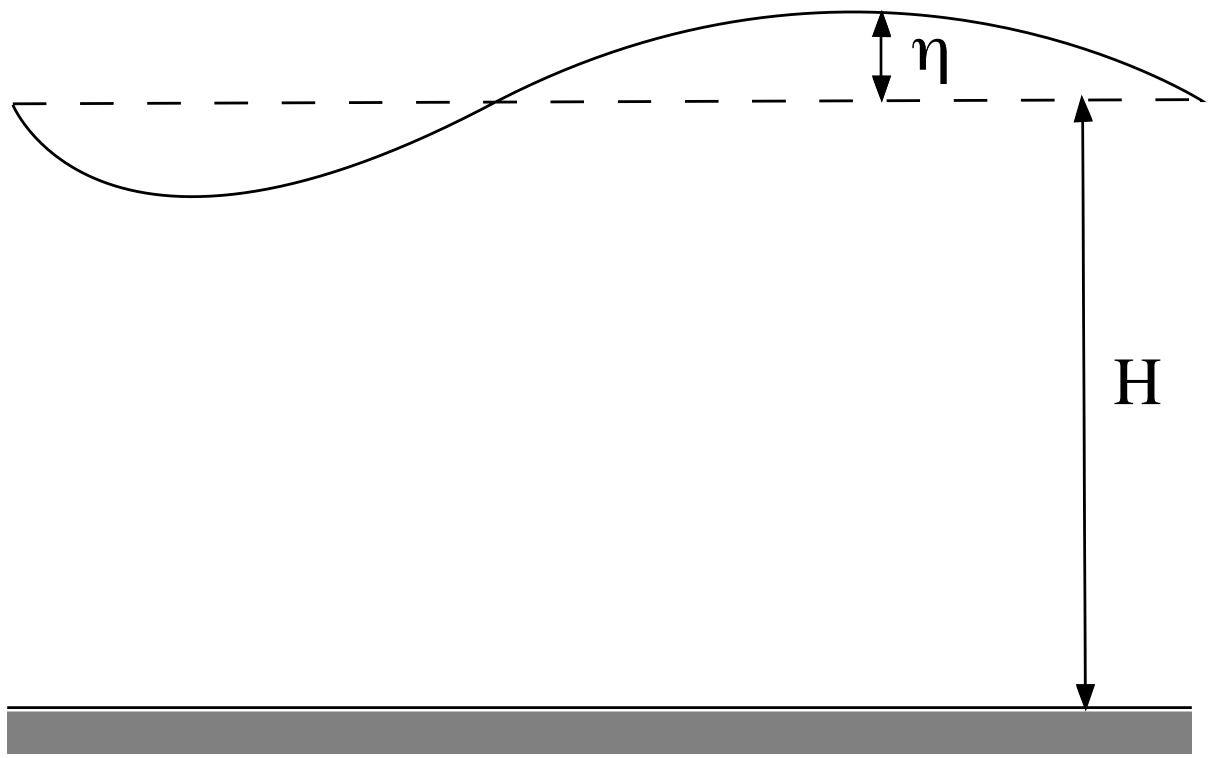

Figure 3: Shallow water system

3 Shallow water equations

If the fluid is of constant density then the above set of equations can be simplified still further to produce the **shallow water equations**. Shallow water refers to the fact that we are considering motions that have a horizontal scale far in excess of the depth of the fluid (a requirement of the hydrostatic approximation). Even the ocean may be considered 'shallow' for certain purposes.

So let us assume we have a homogeneous fluid, i.e. $\rho =$ constant. (We shall consider the effects of the internal variations of density later.) When the ocean is at rest (no motion) the free surface (air/sea interface) will be horizontal. In this state we will take the free surface to be at $z = 0$ and the fluid to be of constant depth H with the lower boundary at $z = -H$ (fig. 3). The pressure (from (19)) is then only a function depth

$$p = -\rho g z \quad .$$

Here we have assumed the pressure at the free surface is zero (a perfectly valid thing to do so long as the atmopheric pressure is at the sea surface is constant). Vertical displacement of the free surface, given by $\eta(x, y, t)$, will change the pressure at a given point in the fluid since the column of water above the point will increase or decrease in depth dependant on the sign of η. The pressure then becomes

$$p = -\rho g z + \rho g \eta$$

Substituting into the horizontal momentum equations (20) and (21)

$$\frac{Du}{Dt} - fv = -g\frac{\partial\eta}{\partial x} \tag{25}$$
$$\frac{Dv}{Dt} + fu = -g\frac{\partial\eta}{\partial y} \tag{26}$$

There are two things to note from these equations. Firstly variations in the height of the free surface will drive motions in the fluid. Secondly, since the pressure force produced by these height variations are constant with depth then the changes in the flow will also be constant with depth. The implication is that if u, v are independent of z at any time then they will always be.

We require an expression for η. This is got by considering the horizontal divergence of the fluid flow which is

$$\frac{\partial(u(H+\eta))}{\partial x} + \frac{\partial(v(H+\eta))}{\partial y} \quad .$$

If the horizontal divergence is positive (negative) then this must lead to a lowering (raising) of the free surface. Thus

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left(u(H+\eta)\right) + \frac{\partial}{\partial y}\left(v(H+\eta)\right) = 0 \quad . \tag{27}$$

Equations (25-27) form a closed set for the variables u, v and η.

The above equations are non–linear. It is often helpful to consider only small perturbations to the system and hence linearize the equations. The linear form for (25-27) is

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial\eta}{\partial x} \tag{28}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial\eta}{\partial y} \tag{29}$$

$$\frac{\partial\eta}{\partial t} + H\frac{\partial u}{\partial x} + H\frac{\partial v}{\partial y} = 0 \quad . \tag{30}$$