Forcing by surface stress

The wind blowing over the ocean surface exerts a stress on the ocean. This stress represents a retarding force for the atmosphere and a major driving force for the ocean. In order to study the effects of a surface stress on the dynamics of the ocean we need to include the force exerted by a stress in the equations of motion.

Consider a horizontal stress (X, Y). Then if the stress varies with depth within the fluid there will be a net force acting to accelerate the fluid. This force per unit mass is

$$\frac{1}{\rho}\left(\frac{\partial X}{\partial z}, \frac{\partial Y}{\partial z}\right).$$

The horizontal components of the momentum equations become

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \frac{1}{\rho}\frac{\partial X}{\partial z} \tag{1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \frac{1}{\rho}\frac{\partial Y}{\partial z}. \tag{2}$$

Note: We have only included vertical derivatives of the horizontal stresses. This is because of the difference in vertical and horizontal scales in the atmospheric and oceanic boundary layers ($\delta \sim 1000m$ for the atmosphere, $10-100m$ for the ocean). The boundary layer is the region of flow close to a boundary which is directly influenced by the presence of the boundary.

Consider the steady state solution. Away from boundaries stresses are negligible and the equations reduce to the geostrophic relation

$$-fv_g = -\frac{1}{\rho}\frac{\partial p'}{\partial x}; \quad fu_g = -\frac{1}{\rho}\frac{\partial p'}{\partial y}.$$

As the boundary is approached the stresses will become important. Writing the velocity components as

$$u = u_g + u_e; \quad v = v_g + v_e$$

where (u_e, v_e) are the ageostrophic components of velocity then

$$-fv_e = \frac{1}{\rho}\frac{\partial X}{\partial z}; \quad fu_e = \frac{1}{\rho}\frac{\partial Y}{\partial z}. \tag{3}$$

Integrating across the boundary layer gives

$$V_e = \frac{1}{\rho f}X_S; \quad U_e = -\frac{1}{\rho f}Y_S \tag{4}$$

where the boundary is below, (X_S, Y_S) is the surface stress and

$$U_e, V_e = \int_o^\infty u_e, v_e \, dz.$$

The mass transport is at right angles to the direction of the surface stress. We define (U_e, V_e) as the **Ekman volume transport** and $(\rho U_e, \rho V_e)$ as the **Ekman mass transport**.

If the boundary is above

$$V_e = -\frac{1}{\rho f} X_S; \;\; U_e = \frac{1}{\rho f} Y_S. \tag{5}$$

The total Ekman mass transport in the atmosphere and ocean across the air/sea interface is zero.
Note: The Ekman transport is in addition to the geostrophic transport caused by the associated pressure gradient.

A brief discourse on turbulence

To investigate the vertical structure of the Ekman layer (the region in which the ageostrophic flow caused by the surface stress is significant) we need to relate the stress vector (X, Y) to the velocity field (u, v).

For a **laminar** Newtonian flow

$$(X, Y) = \left(\mu \frac{\partial u}{\partial z}, \mu \frac{\partial v}{\partial z} \right)$$

where μ is the viscosity of the fluid. This leads to a $\nu \nabla^2 \underline{u}$ term in the momentum equation $(\nu = \mu/\rho)$. But geophysical boundary layers are turbulent. In most instances we are interested in the 'mean' properties of the flow rather than the details of individual turbulent eddies. However we do need to consider the effect of the 'turbulent' part of the flow on the 'mean'.

To proceed we split the velocity field into two parts: $< \underline{u} >$ to represent the large–scale flow we wish to describe in detail and $\underline{u}'$ representing smaller scale turbulence. Thus

$$\underline{u} = < \underline{u} > + \underline{u}'$$

where $<>$ represents some averaging process (time, space, ensemble).

Note that $< \underline{u}' > = 0$ and $< < \underline{u} > + \underline{u}' > = < \underline{u} >$.

Substituting into the u momentum equation and averaging we obtain

$$\frac{\partial < u >}{\partial t} + < u > \frac{\partial < u >}{\partial x} + < v > \frac{\partial < u >}{\partial y} + < w > \frac{\partial < u >}{\partial z} - f < v > =$$

$$-\frac{1}{\rho}\frac{\partial < p >}{\partial x} + \nu\nabla^2 < u > - \frac{\partial}{\partial x} < u'u' > - \frac{\partial}{\partial y} < v'u' > - \frac{\partial}{\partial z} < w'u' > . \qquad (6)$$

The terms

$$\tau_{ij} = -\rho < u_i u_j > \quad (i = x, y, z)$$

are referred to as the **Reynolds stresses** and represent the actions of the turbulent motions on the mean flow. Averaging the equations introduces the closure problem in that now we have more unknowns than equations.

We will use the simplest closure and relate the Reynolds stresses to the gradient of the mean flow, e.g.

$$\begin{aligned}
\frac{\tau_{xx}}{\rho} &= - < u'u' > = & 2A_h\frac{\partial < u >}{\partial x} \\
\frac{\tau_{xy}}{\rho} &= - < u'v' > = & A_h\left(\frac{\partial < u >}{\partial y} + \frac{\partial < v >}{\partial x}\right) \\
\frac{\tau_{xz}}{\rho} &= - < u'w' > = & A_v\frac{\partial < u >}{\partial z} + A_H\frac{\partial < w >}{\partial x}
\end{aligned}$$

A_h and A_v are called the horizontal and vertical eddy viscosity coefficients, respectively, and here assumed to be constant.

Then

$$\frac{\partial < u >}{\partial t} + < u > \frac{\partial < u >}{\partial x} + < v > \frac{\partial < u >}{\partial y} + < w > \frac{\partial < u >}{\partial z} - f < v > =$$

$$-\frac{1}{\rho}\frac{\partial < p >}{\partial x} + \nu\nabla^2 < u > + A_h\left(\frac{\partial^2 < u >}{\partial x^2} + \frac{\partial^2 < u >}{\partial y^2}\right) + A_v\frac{\partial^2 < u >}{\partial z^2}. \qquad (7)$$

From now on we will drop the <>'s and assume we are dealing with mean quantities only.

The values of the eddy viscosities, A_h and A_v, will scale on $u^*\mathcal{L}$, where u^* and $\mathcal{L}$ are typical velocity and length scales of the turbulence. We therefore exoect the value of the eddy viscosity to be dependent on the type of flow we are considering.

Within the boundary layers of the atmosphere and ocean the turbulence is three dimensional and $A_h \simeq A_v$.

Typical values for the atmospheric b.l. ~ 10 m^{2}s^{-1}
For the oceanic b.l. $\sim 10^{-2}$ m^{2}s^{-1}.

Outside the b.l. the flow is much more anisotropic.

For the ocean $A_v \sim 10^{-3}$ m^{2}s^{-1}, whilst if we include mesoscale eddies in our definition of the turbulence $A_h \sim 10^2 - 10^4$ m^{2}s^{-1}.

CAUTION: The above is a very simplistic view of the complex nature of turbulent flow. There are many notable examples where the use of an eddy viscosity to describe the affect of turbulent motions on the mean flow is totally inappropriate.

Reference: read Tennekes and Lumley (1972) A first course in turbulence, MIT Press, for a more detailed description of turbulent/mean flow interaction.

Ekman layers

Applying the above to the Ekamn flow in the ocean close to the air/sea interface then (3) becomes

$$-fv_e = A_v \frac{\partial^2 u_e}{\partial z^2}; \quad fu_e = A_v \frac{\partial^2 v_e}{\partial z^2} \tag{8}$$

Note we have ignored the contribution from molecular viscosity ($\nu \simeq 10^{-6}\ \mathrm{m^2 s^{-1}}$) and the gradient of horizontal stresses. Also we have assumed the vertical shear of the geostrophic velocity (u_g, v_g) is small compared with that of (u_e, v_e).

Let the ocean upper surface be at $z = 0$, then the boundary conditions are

$$(u_e, v_e) \rightarrow 0 \text{ as } z \rightarrow -\infty$$

and

$$A_v \left(\frac{\partial u_e}{\partial z}, \frac{\partial v_e}{\partial z} \right) = \frac{1}{\rho}(X_s, Y_s) \text{ on } z = 0$$

Method of solution

Combine equations (8) to give

$$if(u_e + iv_e) = A_v \frac{\partial^2}{\partial z^2}(u_e + iv_e)$$

Thus

$$u_e + iv_e \sim \exp\left[\left(\frac{if}{A_v} \right)^{\frac{1}{2}} z \right]$$

or

$$u_e + iv_e = Ae^{(1+i)\left(\frac{f}{2A_v}\right)^{1/2} z} + Be^{-(1+i)\left(\frac{f}{2A_v}\right)^{1/2} z}$$

Applying the boundary conditions

$$B = 0$$

$$\frac{X_s}{\rho A_v} + \frac{Y_s}{\rho A_v} = A(1+i)\left(\frac{f}{2A_v}\right)^{\frac{1}{2}}$$

Then

$$u_e = \frac{e^{\left(\frac{f}{2A_v}\right)^{1/2} z}}{(2A_v f)^{1/2}} \left[\left(\frac{X_s}{\rho} + \frac{Y_S}{\rho}\right)\cos\left(\frac{f}{2A_v}\right)^{\frac{1}{2}} z + \left(\frac{X_s}{\rho} - \frac{Y_S}{\rho}\right)\sin\left(\frac{f}{2A_v}\right)^{\frac{1}{2}} z\right] \quad (9)$$

$$v_e = \frac{e^{\left(\frac{f}{2A_v}\right)^{1/2} z}}{(2A_v f)^{1/2}} \left[\left(\frac{X_s}{\rho} + \frac{Y_S}{\rho}\right)\sin\left(\frac{f}{2A_v}\right)^{\frac{1}{2}} z - \left(\frac{X_s}{\rho} - \frac{Y_S}{\rho}\right)\cos\left(\frac{f}{2A_v}\right)^{\frac{1}{2}} z\right] \quad (10)$$

At the surface $z = 0$ (with $Y_S = 0$)

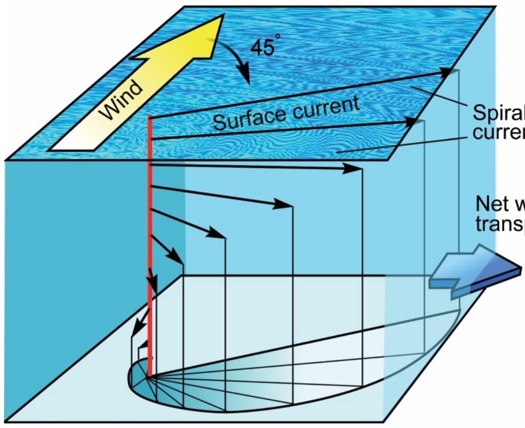
$$u_e = \frac{X_S}{\rho(2A_v f)^{1/2}};\ v_e = -\frac{X_S}{\rho(2A_v f)^{1/2}} \quad (11)$$

i.e. the flow in the ocean at the surface is at 45^o to the right of the surface stress. The stress induced flow diminshes with depth on a scale

$$\delta = \left(\frac{2A_v}{f}\right)^{\frac{1}{2}}$$

For the ocean, with $A_v = 10^{-2}$ m^2s^{-1}, then $\delta \simeq 15m$. **The direct influence of the wind is confined to a shallow depth.**

The direction of flow changes with height – the **Ekman spiral**



A similar solution can be obtained for the Ekamn spiral in the atmosphere. With a geostrophic flow, (u_g, v_g), well away from the surface the boundary conditions are now

$$(u_e, v_e) \to 0 \text{ as } z \to \infty$$

$$(u_e, v_e) = -(u_g, v_g) \text{ on } z = 0.$$

It can be shown that

$$\frac{X_S}{\rho} = \left(\frac{fA_v}{2}\right)^{1/2} (u_g - v_g) \tag{12}$$

$$\frac{Y_S}{\rho} = \left(\frac{fA_v}{2}\right)^{1/2} (u_g + v_g) \tag{13}$$

i.e. the surface stress is at 45^o to the left of the geostrophic flow aloft (N. hemisphere).

Observations show that this angle is between $0^o - 40^o$ depending on whether the flow is stably or unstably stratified. It is $\sim 20^o$ for neutral conditions. (A more sophisticated turbulence model gives better agreement with observation.)

Ekman pumping

Variations in the stress on a surface leads to horizontal convergences and divergences of the Ekman transport which is compensated for by a vertical velocity divergence. This process is called **Ekman pumping**.

Integrating the continuity equation with respect to z we get

$$\frac{\partial U_e}{\partial x} + \frac{\partial V_e}{\partial y} - w_e = 0$$

where w_e is the vertical velocity at the base of the Ekman layer and the boundary is above.

In terms of the stress

$$w_e = \frac{1}{f}\left(\frac{\partial Y_S}{\partial x} - \frac{\partial X_S}{\partial y}\right) \tag{14}$$

(assuming the wind varies more rapidly than f.) I.e. w_e is proportional to the curl of the wind stress. Note that so long as we know the surface stress this result is independent of the turbulence closure.

A similar expression is valid for the atmosphere (the Ekman pumping velocity is the same in both the atmosphere and ocean)

For the atmospheric boundary layer (and the ocean bottom boundary layer) we can write the Ekman pumping velocity in terms of the geostrophic velocity. Thus

$$w_e = \left(\frac{fA_v}{2}\right)^{1/2}\left(\frac{\partial v_g}{\partial x} - \frac{\partial u_e}{\partial y}\right) \tag{15}$$

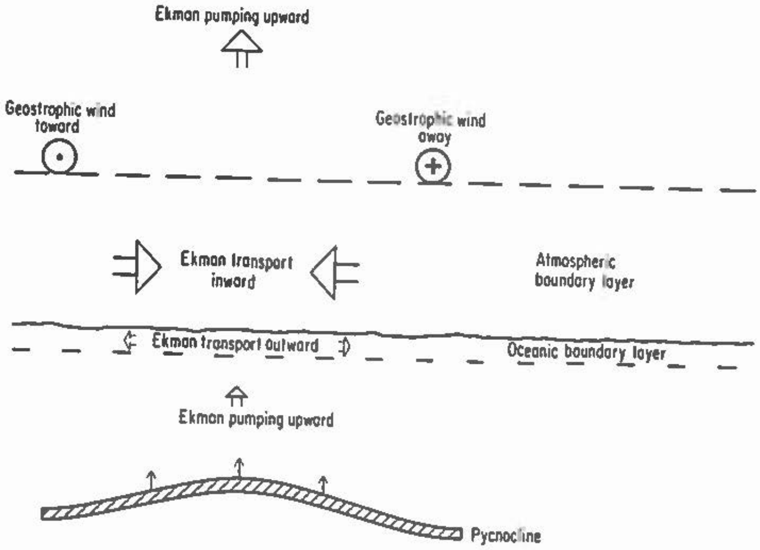


Fig. 9.4. Section through a cyclone over the ocean showing the adjustments due to Ekman transports. The geostrophic wind gives, as shown, a cyclonic rotation around the low-pressure center. Consequently, the Ekman transport in the atmospheric boundary layer is inward, bringing mass in to "fill" the low, and the associated vertical "pumping" velocity is therefore upward. The Ekman mass transport in the oceanic boundary layer is equal and opposite to that in the atmosphere, so there is an outward mass transport and upward pumping velocity in the ocean. This tends to raise the thermocline and create a low-pressure center in the ocean.

Gill, 1982