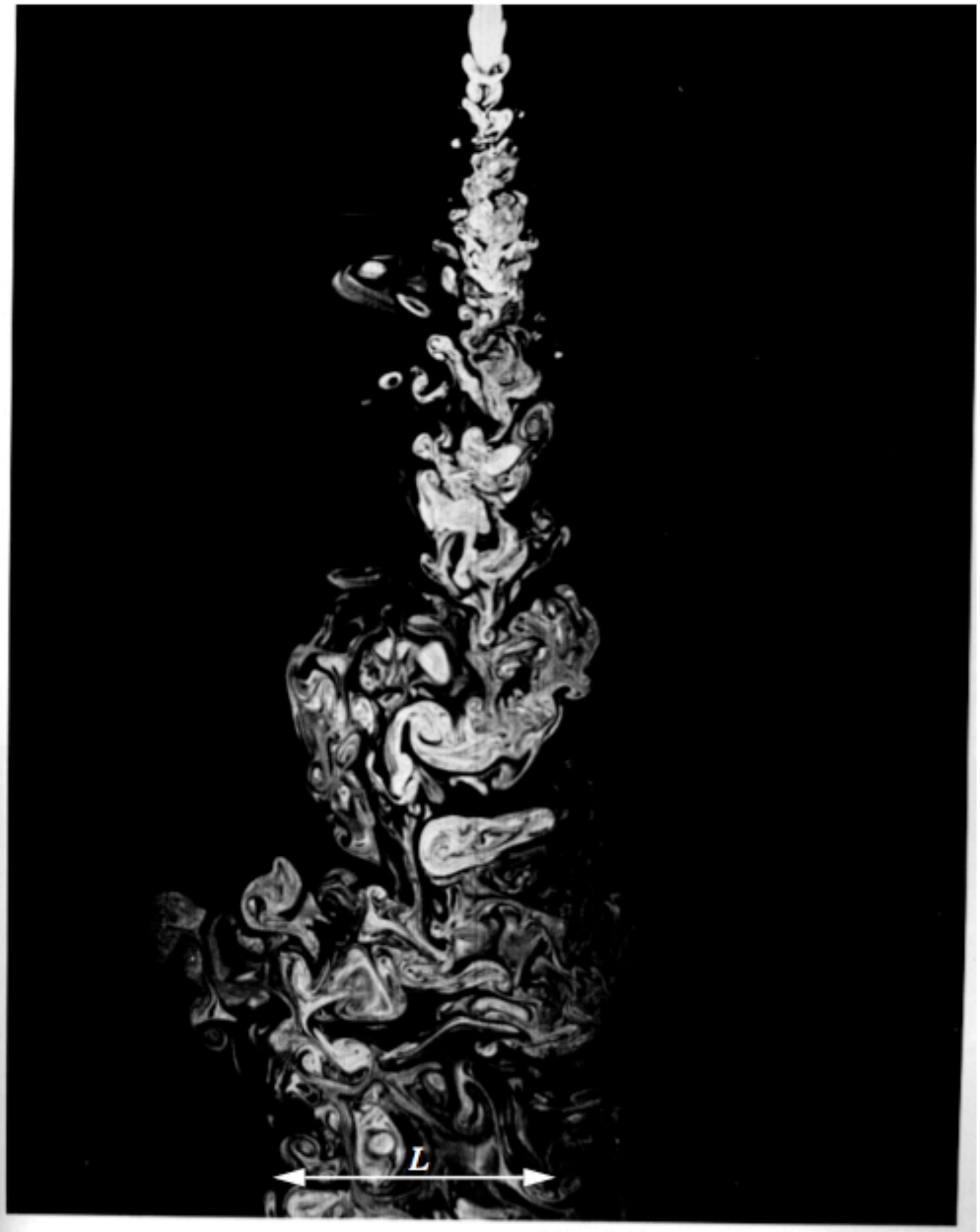
Lecture 2. Turbulent Flow



166. Turbulent water jet. Laser-induced fluorescence shows the concentration of jet fluid in the plane of symmetry of an axisymmetric jet of water directed downward into water. The Reynolds number is approximately 2300. The spatial resolution is adequate to resolve the Kolmogorov scale in the downstream half of the photograph. *Dimotakis, Lye & Papantoniou 1981*

Note the diverse scales of eddy motion and self-similar appearance at different lengthscales of this turbulent water jet. If L is the size of the largest eddies, only very small eddies of size $L\,\mathrm{Re}^{-3/4}$ (the *Kolmogorov scale*) experience substantial viscous dissipation.

In this lecture…

- What is turbulence?
- How do we statistically quantify it?
- Turbulent energy cascade and Fourier spectrum.

Description of Turbulence

Turbulence is characterized by disordered, eddying fluid motions over a wide range of lengthscales. While turbulent flows still obey the deterministic equations of fluid motion, a small initial perturbation to a turbulent flow rapidly grows to affect the entire flow (loss of predictability), even if the external boundary conditions such as pressure gradients or surface fluxes are unchanged. We can imagine an infinite family or *ensemble* of turbulent flows all forced by the same boundary conditions, but starting from a random set of initial flows. One way to create such an ensemble is by adding random small perturbations to the same initial flow, then looking at the resulting flows at a much later time when they have become decorrelated with each other.

Turbulent flows are best characterized statistically through **ensemble averaging**, i. e. averaging some quantity of interest across the entire ensemble of flows. By definition, we cannot actually measure an ensemble average, but turbulent flows vary randomly in time and (along directions of symmetry) in space, so a sufficiently long time or space average is usually a good approximation to the ensemble average. Any quantity a (which may depend on location or time) can be partitioned

$$a = \overline{a} + a' ,$$

where $\overline{a}$ is the **ensemble mean** of a, and a' is the fluctuating part or perturbation of a. The ensemble mean of a' is zero by definition; a' can be characterized by a probability distribution whose spread is characterized by the **variance** $\overline{a'a'}$. One commonly referred to measure of this type is the turbulent kinetic energy (TKE) per unit mass, often represented by the symbol e:

$$\text{TKE} \quad e = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right).$$

This is proportional to the variance of the magnitude of the velocity perturbation:

$$\text{TKE} = \frac{1}{2}\overline{q'q'} \;, \quad q' = (u'^2 + v'^2 + w'^2)^{1/2}.$$

We may also be interested in **covariances** between two quantities a and b. These might be the same field measured at different locations or times (i. e., the spatial or temporal autocorrelation), or different fields measured at the same place and time. For instance, the upward eddy heat flux is proportional to the covariance $\overline{w'T'}$ between vertical velocity w and temperature T). Variances and covariances are called **second-order moments** of the turbulent flow. These take a longer set of measurements to determine reliably than ensemble means.

The temporal autocorrelation of a perturbation quantity a' measured at a fixed position,

$$R(T) = \frac{\overline{a'(t)a'(t+T)}}{\overline{a'(t)a'(t)}}$$

can be used to define an **integral time scale**

$$\tau_a = \int_0^\infty R(T)\,dT$$

which characterizes the timescale over which perturbations of *a* are correlated. One may similarly define an integral length scale.

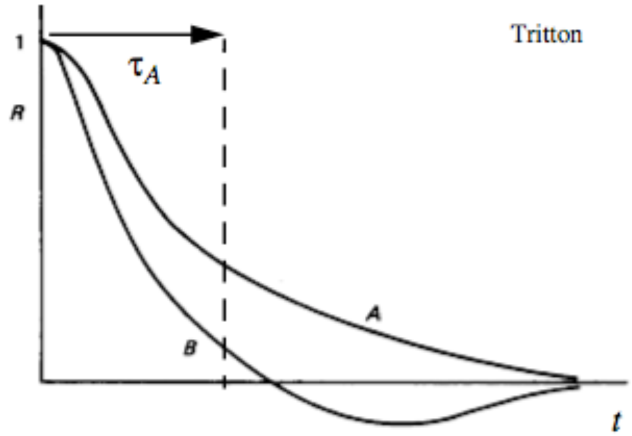


Figure 20.5 Typical correlation curves.

One commonly referred-to statistic for turbulence in which buoyancy forces are important involves third-order moments. The vertical velocity **skewness** is defined

$$S = \frac{\overline{w'w'w'}}{\overline{w'w'}^{3/2}}.$$

The skewness is positive where perturbation updrafts tend to be more intense and narrower than perturbation downdrafts, e. g. in cumulus convection, and is negative where the downdrafts are more intense and narrower, e. g. at the top of a stratocumulus cloud. Skewness larger than 1 indicates quite noticeable asymmetry between perturbation up and downdrafts. For example, consider an ideal flow with uniform updrafts covering a fraction *A* of a domain of width *L* and downdrafts covering a fraction 1-*A* of the domain. Mass conservation implies an equal domain-averaged flux *M* of air going up vs. down:

$$w' = \begin{cases} M/A, & 0 < x/L < A \\ -M/(1-A), & A < x/L < 1 \end{cases}$$

Simple calculation shows

$$\overline{w'w'} = M^2\left\{A\left(\frac{1}{A}\right)^2 + (1-A)\left(-\frac{1}{1-A}\right)^2\right\} = \frac{M^2}{A(1-A)}$$

$$\overline{w'w'w'} = M^3\left\{A\left(\frac{1}{A}\right)^3 + (1-A)\left(-\frac{1}{1-A}\right)^3\right\} = \frac{M^2(1-2A)}{A^2(1-A)^2}$$

$$S = \frac{1-2A}{A^{1/2}(1-A)^{1/2}}$$

The skewness is zero if updrafts and downdrafts are symmetric (A = ½). As the updraft area fraction *A* goes down, the skewness goes up.

Fourier spectra in space or time of perturbations are commonly used to help characterize the distribution of the fluctuations over different length and time scales. For example, given a long time series of a quantity $a(t)$, we can take its Fourier transform with respect to frequency ω over an arbitrary time interval of any length τ:

$$\tilde{a}(\omega_n) = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} a'(t)\exp(-i\omega_n t)\,dt, \quad \omega_n = 2\pi n/\tau, \quad n = 0, \pm 1, \pm 2, \ldots$$

The temporal **power spectrum** of a is $\overline{\left|\tilde{a}(\omega_n)\right|^2}$; note this is an ensemble mean that must be estimated using multiple realizations or multiple independent time segments of $a(t)$. If we let the sampling interval $\tau \to \infty$, the power spectrum becomes a continuous function $\tilde{S}_a(\omega)$ of ω. The power spectrum can be shown to be the Fourier transform of the autocovariance:

$$\tilde{S}_a(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{a'(t)a'(t+T)}\exp(-i\omega T)\,dT . \quad \text{(this is real and positive for all } \omega\text{)}$$

Conversely, given the power spectrum, one can recover the autocovariance by an inverse Fourier transform, and in particular, the variance is the integral of the power spectrum over frequency,

$$\overline{a'(t)a'(t)} = \int_{-\infty}^{\infty} \tilde{S}_a(\omega)\,d\omega ,$$

so we can think of the power spectrum as a partitioning of the variance of a between frequencies.

For spatially **homogeneous** turbulence one can do a 3D Fourier transform of the spatial auto-covariance function to obtain the spatial power spectrum vs. wavevector $\mathbf{k}$,

$$\hat{S}_a(\mathbf{k}) = \frac{1}{(2\pi)^3}\iiint_{\text{all }\mathbf{R}} \overline{a'(\mathbf{r})a'(\mathbf{r}+\mathbf{R})}\exp(-i\mathbf{k}\cdot\mathbf{R})\,d\mathbf{R};$$

again the variance a is the integral of the power spectrum over all wavevectors,

$$\overline{a'(\mathbf{r})a'(\mathbf{r})} = \iiint_{\text{all }\mathbf{k}} \hat{S}_a(\mathbf{k})\,d\mathbf{k}.$$

If the turbulence is also **isotropic**, i. e. looks the same from all orientations, then the power spectrum depends only on the magnitude k of the wavevector, which is called the **wavenumber**. Then, we can partition the variance into different wavenumber bands:

$$\overline{a'(\mathbf{r})a'(\mathbf{r})} = \int_0^{\infty} \hat{S}_a(k)4\pi k^2\,dk.$$

In particular, for homogeneous isotropic turbulence we can partition TKE into contributions from all wavenumbers; this is called the **energy spectrum** $E(k)$.

$$\text{TKE} = \frac{1}{2}\overline{q'(\mathbf{r})q'(\mathbf{r})} = \int_0^{\infty} E(k)\,dk .$$

Roughly speaking, the energy spectrum at a particular wavenumber k can be visualized as being due to eddies whose characteristic diameter l is the corresponding half-wavelength π/k. The typical overturning velocity V_l of such eddies can be roughly estimated by integrating the energy spectrum over an ‘octave’ (power of two in wavenumber) centered on k:

$$V_l^2 / 2 \sim \int_{2^{-1/2}\pi/l}^{2^{1/2}\pi/l} E(k)\,dk$$

Turbulent Energy Cascade (see Vallis, *Atmos. & Ocean Fluid Dyn.*, Chapter 8)

Ultimately, boundary layer turbulence is due to continuous forcing of the mean flow toward a state in which shear or convective instabilities grow. These instabilities typically feed energy mostly into eddies whose characteristic size is comparable to the boundary layer depth. When these eddies become turbulent, considerable variability is also seen on much smaller scales. This is often described as an **energy cascade** from larger to smaller scales through the interaction of eddies. It is called a cascade because eddies are deformed and folded most efficiently by other eddies of comparable scales, and this squeezing and stretching transfers energy between nearby lengthscales. Thus the large eddies feed energy into smaller ones, and so on until the eddies become so small as to be viscously dissipated. There is typically a range of eddy scales larger than this at which buoyancy or shear of the mean flow are insignificant to the eddy statistics compared to the effects of other turbulent eddies; in this **inertial subrange** of scales the turbulent motions are roughly homogeneous, isotropic, and inviscid, and if fact from a photograph one could not tell at what lengthscale one is looking, i. e. the turbulence is self-similar.

Garratt

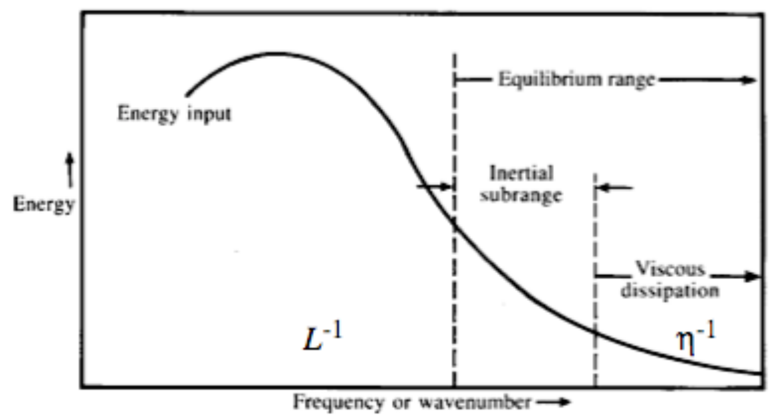


Fig. 2.1 Schematic representation of the energy spectrum of turbulence.

Dimensional arguments have always played a central role in our understanding of turbulence due to the complexity and self-similarity of turbulent flow. Kolmogorov (1941) postulated that for large Reynolds number, the statistical properties of turbulence above the viscous dissipation scale are independent of viscosity and depend only on the rate at which energy produced at the largest scale L is cascaded down to smaller eddies and ultimately dissipated by viscosity. This is measured by the average energy dissipation rate ε per unit mass (units of energy per unit mass per unit time, or $m^2\ s^{-3}$). If the largest scale eddies have characteristic eddy velocity V_L, dimensional analysis implies

$$\varepsilon \propto V_L^{\ 3}/L,$$

and the dissipation timescale is the eddy turnover timescale L/V_L, which is typically O(1000 m/1 m s^{-1}) = 1000 s in the ABL. This means that if its large-scale energy source is cut off, turbulence decays within a few turnover times. The viscous dissipation lengthscale or **Kolmogorov scale** η depends on ε (m^2s^{-3}) and ν (m^2s^{-1}) , so dimensionally

$$\eta = (\nu^3/\varepsilon)^{1/4} \text{ (1 mm for the ABL) } = Re^{-3/4} L.$$

Kolmogorov argued that the energy spectrum $E(k)$ within the inertial subrange can depend only on the lengthscale, measured by wavenumber k, and ε. Noting that $E(k)$ has units of TKE/wavenumber = $m^2s^{-2}/m^{-1} = m^3s^{-2}$, dimensional analysis implies the famous **-5/3 power law**,

$$E(k) \propto \varepsilon^{2/3} k^{-5/3}, \qquad L^{-1} << k << \eta^{-1}.$$

If V_l is the typical turnover velocity of eddies of diameter l in the inertial range, we can argue from the Kolmogorov spectrum or from dimensional arguments that

$$V_l^3/l \propto \varepsilon, \quad \text{so} \quad V_l^3/l \propto \varepsilon \propto V_L^{\,3}/L \quad \text{and} \quad V_l/V_L = (l/L)^{1/3}$$

Similarly, the spatial power spectra of velocity components and scalars a also follow $\hat{S}_a(k) \propto k^{-5/3}$ in the inertial range. This follows from assuming a constant turbulent dissipation rate χ of scalar variance (Vallis 8.5). If the scalar power spectrum is proportional to χ and is also assumed to depend on ε and k, dimensional analysis gives the desired power law.

The spatial power spectrum can be measured in one direction by a sensor moving with respect to the boundary layer at a speed U comparable to or larger than V_L, i. e. if the wind is blowing different turbulent eddies past a sensor on the ground, or if we take measurements from an aircraft. We must invoke **Taylor's ('frozen turbulence') hypothesis** that the statistics of the turbulent field are similar to what we would measure if the turbulent field remained unchanged and just advected by at the mean wind speed U. In general, empirically this appears to be a good assumption. Temporal power spectra $\tilde{S}_a(\omega)$ gathered in this way can be converted to spatial power spectra by substituting $\omega = Uk$,

$$\hat{S}_a(k) = U\tilde{S}_a(Uk) \quad \text{for turbulence moving by with mean speed } U.$$

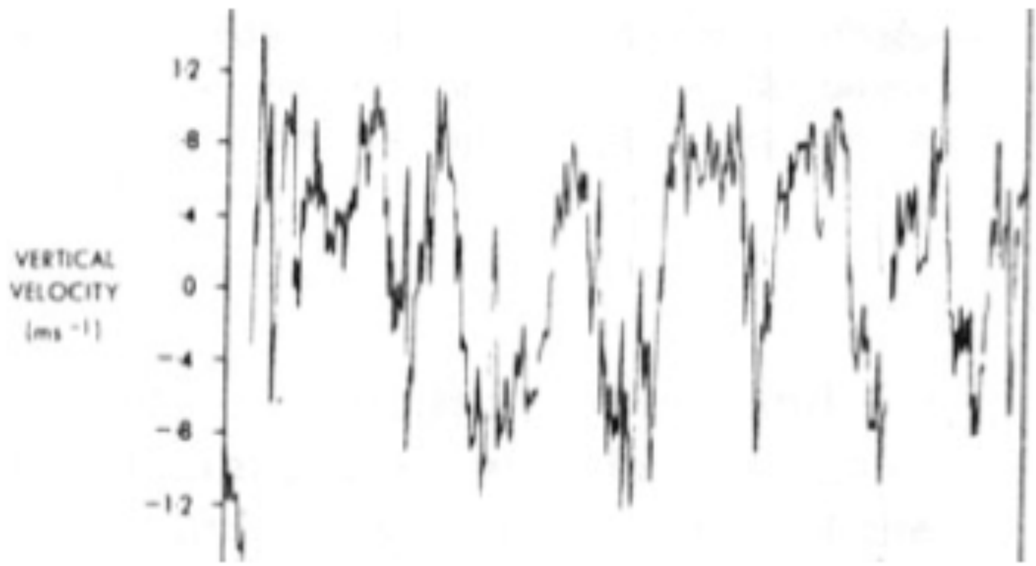
Thus, we expect an $\omega^{-5/3}$ temporal power spectrum for scalars and velocity components in the inertial subrange.

The figures below show measurements from a tethered balloon stationed in a convecting cloud-topped boundary layer at 85% of the inversion height. The mean wind of U = 7 m s^{-1} is considerably larger than the characteristic large-eddy velocity V_L = 1 m s^{-1}, so Taylor's hypothesis is safe. The time series shows up and downdrafts associated with large eddies with width and height L comparable to the BL depth of 1 km, with turbulent fluctuations associated with smaller eddies. The corresponding temporal power spectrum (triangles) is plotted as $\omega\tilde{S}_a(\omega)$. As expected, this has a $\omega^{-2/3}$ dependence in the inertial range, and decays at low frequencies that correspond to lengthscales larger than L.

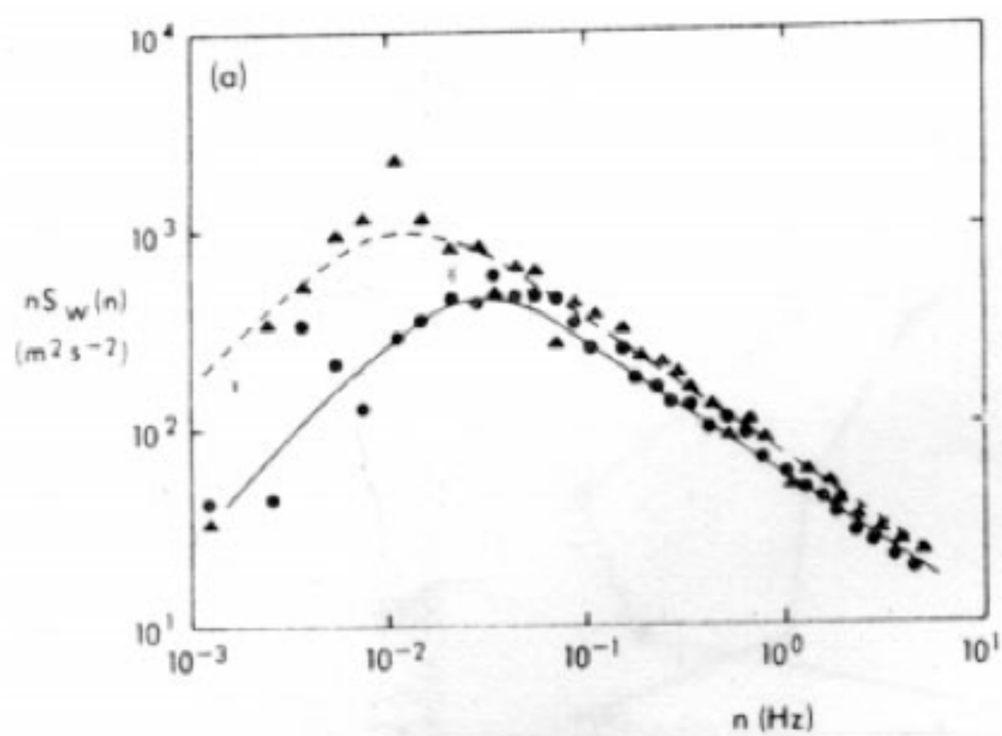
The second spectrum (circles) is in the entrainment zone, which is in a very sharp and strong inversion (stable layer) at the BL top. Here, large scale, strong, vertical motions are suppressed, and the turbulence is highly anisotropic at these scales, but at small scales (a few meters or less) an inertial range is still observed.

Interestingly, 2D 'turbulence' doesn't produce an energy cascade to small scales; instead, in 2D (as simulated on the computer) energy tends to be transferred to the largest scale motions permitted by the boundaries, and broad regions of smoothly varying flow appear, interrupted by shear lines and intense long-lived vortices.

$E(k) \sim k^{-3}$



Vertical velocity trace over a 10 minute period, corresponding to an advection distance of 4200 m, in a 1 km deep convecting boundary layer



Temporal power spectrum of vertical velocity. Triangles correspond to height shown above, and circles are in the entrainment zone at BL top.