# 7

## TURBULENT TRANSPORT

As a turbulent flow moves, it carries fluid from place to place. A tiny parcel of fluid (small, say, compared to the Kolmogorov microscale, but large compared to molecular scales) gradually wanders away from its initial location. This is the mechanism that is responsible for the large transfer rates observed in turbulent flows. In the preceding chapters, the transport capability of turbulence was represented by such quantities as the momentum flux $-\overline{\rho uv}$ and the heat flux $-\rho c_p \overline{\theta v}$; estimates for these were obtained by similarity arguments and dimensional reasoning. Here, we study the details of the process of transport. We first analyze how turbulent motion transports fluid points; then, in the second half of this chapter, we deal with the transport (dispersion, mixing) of contaminants.

## 7.1
### Transport in stationary, homogeneous turbulence
We would like to be able to predict transport in real flows, which generally are inhomogeneous and nonstationary. This is the heart of the turbulence problem; unfortunately, it is impossible to describe the details of transport in other than very simple cases. Let us first discuss the motion of a single fluid "point" in stationary, homogeneous turbulence without mean velocity. This is an idealized situation, because turbulence without a mean velocity gradient has no source of energy, so that it decays and cannot be stationary. More important, this idealized case may not even be relevant to transport in real decaying flows, because (as we later see) the "memory time" of a fluid point is usually of the order of the decay time, so that a real decaying flow never appears even approximately stationary to a wandering point. Consequently, we have to be careful in generalizing the conclusions we obtain for this idealized turbulence; we should not be surprised if the conclusions have qualitative significance only.

**Stationarity**   Before we start the analysis, let us ask when we may expect the velocity of a wandering point to be a stationary (statistically steady) function of time. This question, of course, bears on the applicability of the central limit theorem (Section 6.5). Clearly, it is necessary that the flow be stationary itself. If the flow is also homogeneous, we are assured that the velocity of the wandering point is stationary. This case is discussed first. If the flow is

not homogeneous and unbounded in the direction of inhomogeneity, the moving point wanders into regions of progressively different characteristics. For example, in a boundary layer the flow is distinctly inhomogeneous in the cross-stream direction. As time proceeds, the boundary layer grows and a wandering point moves progressively farther away from the wall into regions where the turbulence properties are different. In such a case, the velocity of a wandering point is not stationary. In a pipe flow, on the other hand, the flow is homogeneous in the streamwise direction and inhomogeneous, but also bounded, in the cross-stream direction. A wandering point may then move toward one wall, but it eventually returns and moves toward the other. Hence, we expect its velocity to be stationary. We conclude that the velocity of a wandering point is stationary if the flow is stationary and bounded in all directions of inhomogeneity.

**Stationary, homogeneous turbulence without mean velocity**  Let us analyze the motion of a fluid point in stationary, homogeneous turbulence without mean velocity (Figure 7.1). The velocity at time $t$ of a moving point which was at the point $x_i = a_i$ at $t = 0$ will be called $v_i(a, t)$. The use of vector notation (denoted by boldface letters) in the argument of $v_i$ prevents confusion of indices. As we discussed above, $v_i(a, t)$ is a stationary (statistically steady) function; it is called a *Lagrangian velocity.*

The position of the wandering point is the integral of its velocity:

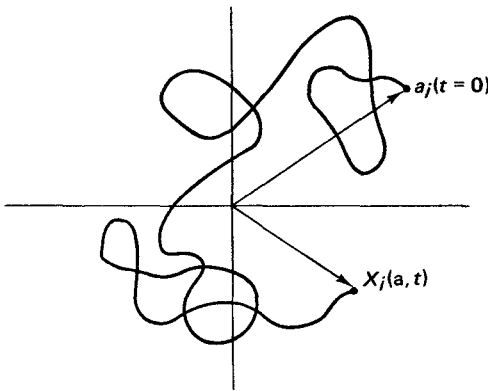$$X_i(a, t) = a_i + \int_0^t v_i(a, t')\, dt',  \tag{7.1.1}$$



Figure 7.1.  The motion of a wandering point

where $X_i(a, 0) = a_i$. The Lagrangian position is $X_i$, and the Eulerian position is $x_i$. The velocity of the moving point is equal to the velocity of the fluid at the point where it happens to be. The velocity $u_i(x, t)$ measured at the location $x_i$ at time $t$ is called the *Eulerian velocity*; it is related to $v_i$ by

$$v_i(a, t) = u_i(X(a, t), t). \tag{7.1.2}$$

The study of transport is very difficult because of (7.1.2). Eulerian velocities $(u_i)$ can be measured by putting a fixed probe in the fluid, but the measurement of Lagrangian velocities $(v_i)$ requires that the motion of "tagged" fluid points be followed with photographic or radioactive tracer techniques. Often, only Eulerian measurements are made; however, the statistics of $u_i$ are not related to those of $v_i$ in a simple way. The problem is that one needs to know $v_i$ in order to find $X_i$ in order to find $u_i$. The problem is similar to that of the passage of light through air with turbulent fluctuations in the index of refraction $n$. The path of a light ray depends on the fluctuations in the $n$ it sees. The path tends to curve around regions with high $n$ and tends to veer away from regions with low $n$, so that the statistics of $n$ experienced by the light ray are different from those seen on a straight line through the turbulent air.

However, because $v_i$ is a stationary function presumably having nonzero integral scales the central limit theorem (Section 6.5) can be applied to the integral (7.1.1). Consider one component of $X_i$–$a_i$, and call this $X_\alpha$–$a_\alpha$. Here, $\alpha$ may be equal to 1, 2, or 3, but we stipulate that the index summation convention does not apply to the index $\alpha$. Because $v_i$ is stationary, $X_\alpha$–$a_\alpha$ asymptotically has a Gaussian probability density; its variance is given by (Taylor, 1921)

$$\overline{(X_\alpha - a_\alpha)^2} = 2\,\overline{v_\alpha^2}\,t\,\int_0^t \left(1 - \frac{\tau}{t}\right)\rho_{\alpha\alpha}(\tau)\,d\tau \cong 2\,\overline{v_\alpha^2}\,t\,\mathscr{T}_{\alpha\alpha}. \tag{7.1.3}$$

The *Lagrangian autocorrelation coefficient* $\rho_{\alpha\alpha}(\tau)$ is defined by

$$\overline{v_\alpha^2}\,\rho_{\alpha\alpha}(\tau) \equiv \overline{v_\alpha(a, t)\,v_\alpha(a, t + \tau)}. \tag{7.1.4}$$

The integral scale of $\rho_{\alpha\alpha}$ is $\mathscr{T}_{\alpha\alpha}$; it is called the *Lagrangian integral scale*. The shape of $\rho_{\alpha\alpha}$ looks approximately like the curve in Figure 6.10.

A great deal of effort has been spent in attempts to predict $\mathscr{T}_{\alpha\alpha}$ from Eulerian data, with very little success. A relatively simple prediction is made shortly. We also have to consider the problem of determining $\overline{v_\alpha^2}$.

The set of equations (7.1.1–7.1.4) is also applicable to molecular diffusion

($X_i$ would be the position of a molecule and $v_i$ would be its velocity). A Lagrangian time integral scale for molecular motion in gases is of the order of a few collision times $\xi/a$ ($\xi$ is the mean free path, $a$ is the speed of sound; see Section 2.2). At ordinary temperatures and pressures, the time scales of interest in diffusion problems are much larger than $\xi/a$, so that the asymptotic form of (7.1.3) applies. The dispersion $\overline{(X_\alpha - a_\alpha)^2}$ is then proportional to $t$ and the coefficient $a^2 (\xi/a) \sim a\xi$ is the molecular diffusivity, which is of the same order as the kinematic viscosity $\nu$ (Section 2.2). In turbulence, however, the time span before (7.1.3) reaches its asymptotic form is not too short to be of interest. In fact, by the time the integral reaches the "diffusion limit" the wandering point has usually left the (approximately homogeneous) part of the flow field where it started. Still, the asymptotic form of (7.1.3) is a useful, though rather crude, approximation in many cases of practical interest. Note that the asymptotic form of (7.1.3) is equivalent to assuming that the eddy diffusivity $\overline{v_\alpha^2} \mathcal{T}_{\alpha\alpha}$ is constant.

**The probability density of the Lagrangian velocity**    In order to make use of (7.1.3), we need to know $\overline{v_\alpha^2}$. The easiest way to predict this is to exploit the fact that an incompressible fluid moving in a box always fills the box. This simple-looking statement has surprising consequences. If we want to integrate a quantity over all the moving fluid points in the box, we can integrate either over their present locations (an Eulerian integral) or over their initial locations (a Lagrangian integral). Because the fluid continues to fill the box as it moves around, either way each point is counted only once, so that it is immaterial which integral we take. Suppose $F(\mathbf{x}, t)$ is the function we wish to integrate over the volume $V$ of the box; the integral statement then reads

$$\iiint\limits_V F(\mathbf{X}(\mathbf{a}, t), t) \, da_1 \, da_2 \, da_3 = \iiint\limits_V F(\mathbf{x}, t) \, dx_1 \, dx_2 \, dx_3. \tag{7.1.5}$$

If an incompressible flow is not confined to a box, a similar statement can be made. The only problem is that the integration volume on the left-hand side is not the same as that on the right-hand side. Points that were initially on the boundaries of the volume $V$ move, so that the new boundaries gradually wander away from the original ones. However, if the velocities involved are of order $\mathcal{u}$, the boundaries move a distance of order $\mathcal{u}t$ in a time $t$, so that the volume difference between the new and the old boundaries is of order $\mathcal{u}tL^2$ ($L = V^{1/3}$ is the length scale of the integration volume). The volume

fraction involved is of order $\alpha t/L$, which, at any fixed time interval $t$, can be made as small as desired by making $L$ large enough. Hence, for an unbounded flow the equivalent of (7.1.5) reads

$$\lim_{V \to \infty} \frac{1}{V} \iiint F(X(a, t), t)\, da_1\, da_2\, da_3 = \lim_{V \to \infty} \frac{1}{V} \iiint F(x, t)\, dx_1\, dx_2\, dx_3.$$

$$(7.1.6)$$

Now, let $F(x, t) = \exp[i k \cdot u(x, t)]$. The average value of this gives the characteristic function of the Eulerian velocity field (note that we use vectors $k$ and $u$ here; all three components of $u_i$ are treated simultaneously). On the other hand, after averaging, $F(X(a, t), t)$ gives the characteristic function of the Lagrangian velocity field. Substituting the Eulerian and Lagrangian characteristic functions into (7.1.6) and taking averages, we obtain

$$\lim_{V \to \infty} \overline{\frac{1}{V} \iiint \exp[i k \cdot v(a,t)]\, da_1\, da_2\, da_3}$$

$$= \lim_{V \to \infty} \frac{1}{V} \iiint \overline{\exp[i k \cdot v(a, t)]}\, da_1\, da_2\, da_3$$

$$= \overline{\exp[i k \cdot v(a, t)]}$$

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$$= \lim_{V \to \infty} \frac{1}{V} \iiint \overline{\exp[i k \cdot u(x, t)]}\, dx_1\, dx_2\, dx_3$$

$$= \overline{\exp[i k \cdot u(x, t)]}.$$

$$(7.1.7)$$

The characteristic functions can be removed from under the integrals because the turbulence is homogeneous, so that the characteristic functions are independent of position. We conclude that the characteristic functions, and therefore also the probability densities, of the Lagrangian and Eulerian velocity fields are identical in homogeneous turbulence in an incompressible fluid. This implies that in homogeneous, incompressible flow

$$\overline{v_\alpha^2} = \overline{u_\alpha^2}.$$

$$(7.1.8)$$

Therefore, we do not need to determine $\overline{v_\alpha^2}$ in (7.1.3) by direct methods; a relatively easy measurement of $\overline{u_\alpha^2}$ suffices.

The result (7.1.8) might have been expected, but the method used can also be applied to more complex problems. For example, consider fully developed turbulent pipe flow (Section 5.2). Let us take $F(\mathbf{x}, t) = \tilde{u}_1(\mathbf{x}, t)$, which is the instantaneous total axial velocity in the pipe ($x_1$ is the streamwise direction). Since pipe flow is homogeneous in the $x_1$ direction and bounded in the $x_2, x_3$ plane, application of (7.1.6) gives

$$\frac{1}{\pi r^2} \iint \overline{\tilde{v}_1(\mathbf{a}, t)} \, da_2 \, da_3 = \frac{1}{\pi r^2} \iint \overline{\tilde{u}_1(\mathbf{x}, t)} \, dx_2 \, dx_3. \qquad (7.1.9)$$

We may expect that $\overline{\tilde{v}_1(\mathbf{a}, t)}$ will be homogeneous in a cross section of the pipe if $t$ is large enough, because no matter where a moving point starts from, it eventually wanders all around the cross section. With the usual notation convention, $\overline{\tilde{v}_1} = V_1$ and $\overline{\tilde{u}_1} = U_1$, so that (7.1.9) becomes, for large $t$,

$$V_1 = \frac{1}{\pi r^2} \iint \overline{\tilde{u}_1(\mathbf{x}, t)} \, dx_2 \, dx_3 = \frac{1}{\pi r^2} \iint U_1(\mathbf{x}) \, dx_2 \, dx_3 \equiv U_b. \qquad (7.1.10)$$

The mean axial velocity of a moving fluid point in a pipe is thus equal to the *bulk velocity* $U_b$ of the fluid.

The mean-square fluctuation in the axial Lagrangian velocity is obtained in the same way:

$$\frac{1}{\pi r^2} \iint \overline{(\tilde{v}_1(\mathbf{a}, t) - V_1)^2} \, da_2 \, da_3 = \frac{1}{\pi r^2} \iint \overline{v_1^2} \, da_2 \, da_3$$

$$= \frac{1}{\pi r^2} \iint \overline{[\tilde{u}_1(\mathbf{x}, t) - U_b]^2} \, dx_2 \, dx_3. \qquad (7.1.11)$$

Again, the left-hand-side integrand may be expected to be homogeneous. The right-hand-side integrand is not homogeneous; however, with $\tilde{u}_1 = U_1 + u_1$, we obtain

$$\overline{[\tilde{u}_1(\mathbf{x}, t) - U_b]^2} = \overline{u_1^2} + [U_1(\mathbf{x}) - U_b]^2. \qquad (7.1.12)$$

Hence, (7.1.11) becomes

$$\overline{v_1^2} = \frac{1}{\pi r^2} \iint [\overline{u_1^2} + (U_1 - U_b)^2] \, dx_2 \, dx_3. \qquad (7.1.13)$$

The Lagrangian axial velocity variance thus receives contributions both from the Eulerian velocity variance and from the square of the difference between the Eulerian mean velocity and the bulk velocity. Clearly, as a moving point

wanders around in the pipe, its axial velocity fluctuates not only because the Eulerian velocity fluctuates but also because it wanders from time to time into regions where the mean velocity is different from the bulk velocity. The results (7.1.10, 7.1.13) are used in Section 7.2.

It would be tempting to extend this approach to the determination of the Lagrangian correlation. However, no useful results would evolve, because the analysis would yield Lagrangian space-time correlations, which not only are beyond the scope of this book, but also are relatively poorly understood.

**The Lagrangian integral scale**  The second problem associated with applications of the dispersion formula (7.1.3) is the determination of $\mathscr{T}_{\alpha\alpha}$. From simple dimensional reasoning, we know that the Lagrangian (time) integral scale must be proportional to $\ell/u$ in turbulence with a single length scale $\ell$ and a single velocity scale $u$. In Section 2.3, extremely crude mixing-length arguments were used to show that

$$\nu_T = \frac{1}{2}\frac{d}{dt}\overline{(X_2^2)} = \overline{u_2^2}\mathscr{T}_{22}. \tag{7.1.14}$$

In wakes, the eddy viscosity is given by (Table 4.1)

$$\nu_T \cong 2.8\,u_*\ell_*. \tag{7.1.15}$$

Here, $u_*$ is defined on basis of the Reynolds stress and $\ell_*$ is based on the maximum slope of the mean velocity profile. If we take $u'_2$ (the rms value of $u_2$) to be equal to $u_*$ and if we identify $\ell_*$ with the length $\ell$ defined by $\ell = (u'_2)^3/\epsilon$ ($\epsilon$ is the dissipation rate), we obtain from (7.1.14) and (7.1.15)

$$\mathscr{T}_{22} \cong \ell/2.8\,u'_2. \tag{7.1.16}$$

Now, wakes are the most nearly homogeneous flows we have examined, so that (7.1.15) may be approximately valid for homogeneous turbulence. However, (7.1.14) is known to be incorrect because by the time the "diffusion limit" is valid, wandering points have moved to regions of different properties, even in the nearly homogeneous turbulence of a wake. Therefore, an independent estimate of $\mathscr{T}_{22}$, which does not rely on (7.1.14), would be welcome.

Corrsin (1963a) derived an estimate of $\mathscr{T}_{22}$ from spectral similarity considerations. His analysis is discussed in Section 8.5; the result is

$$\mathscr{T}_{22} \cong \ell/3u'_2. \tag{7.1.17}$$

The good agreement between (7.1.16) and (7.1.17) should not be taken too seriously. If we are honest, all we really can state is that $u_2' \mathscr{T}_{22}/\ell \sim 1$, which we should interpret as "somewhere between $\frac{1}{3}$ and 3". Nevertheless, the estimates (7.1.16) and (7.1.17) are quite successful in practice; the coefficient 3 in (7.1.17) may be regarded as an experimentally determined constant (much like the von Kármán constant).

**The diffusion equation** In homogeneous turbulence, the Lagrangian velocity variance is given by (7.1.8) and the Lagrangian integral scale may be estimated with (7.1.17). The asymptotic form of the diffusion equation (7.1.3) then becomes

$$\overline{(X_\alpha - a_\alpha)^2} = \frac{2\overline{u_\alpha^2}\,\ell t}{3u_\alpha'} = \frac{2u_\alpha'\ell t}{3}. \tag{7.1.18}$$

The length $\ell$ is defined by $\ell = (u'_\alpha)^3/\epsilon$, as was stated before. It is often more convenient to use Eulerian integral scales instead of $\ell$. The analysis in Section 8.5 shows that the relations between $\ell$ and the Eulerian integral scales $L_{11}$ and $L_{22}$ (downstream and cross-stream integral scales, respectively), may be estimated as

$$\mathscr{T}_{22} \cong \frac{1}{3}\frac{\ell}{u_2'} \cong \frac{2}{3}\frac{L_{11}}{u_2'} \cong \frac{4}{3}\frac{L_{22}}{u_2'}. \tag{7.1.19}$$

## 7.2
### Transport in shear flows
The case of homogeneous, stationary turbulence discussed in Section 7.1 is rather unrealistic, because turbulence cannot be maintained without mean shear. In this section, we discuss transport in a uniform shear flow and transport in pipes and channels.

**Uniform shear flow** Consider turbulent flow with uniform mean shear ($\partial U_1/\partial x_2$ = constant). The turbulence will be homogeneous in planes normal to the mean velocity $U_1$; however, Lagrangian velocities are not stationary, because the mean flow has no length scale, so that all length scales slowly grow in the streamwise direction, much as in grid turbulence (see Lumley, in Batchelor and Moffatt, 1970). Nevertheless, the rate of growth of the length scales is fairly slow; we may get a qualitative impression of the effects of

mean shear by assuming that the Eulerian velocity field is homogeneous in all directions, so that the Lagrangian velocity field is stationary.

If the mean flow is defined by

$$U_1 = Sx_2, \quad U_2 = U_3 = 0, \tag{7.2.1}$$

the position of a moving point is given by

$$X_1(a, t) = a_1 + \int_0^t [SX_2(a, t') + v_1(a, t')] \, dt', \tag{7.2.2}$$

$$X_2(a, t) = a_2 + \int_0^t v_2(a, t') \, dt', \tag{7.2.3}$$

$$X_3(a, t) = a_3 + \int_0^t v_3(a, t') \, dt'. \tag{7.2.4}$$

Because the turbulence is stationary and homogeneous, the fluctuating Lagrangian velocities $v_1$, $v_2$, and $v_3$ are stationary. From the central limit theorem we conclude directly that $X_2$ and $X_3$ asymptotically have Gaussian distributions, whose variance is given by (7.1.18). However, the downstream transport has to be determined separately because of the presence of the mean shear $S$. As a wandering point moves in the $x_2$ direction, it moves into a region with a different mean velocity, so that it tends to move faster (or slower, as the case may be) than in a flow without shear.

If the mean value of (7.2.3) is combined with the mean value of (7.2.2), there results

$$\overline{X_1} = a_1 + Sa_2t. \tag{7.2.5}$$

This states that the mean position moves with the mean velocity of the initial location $(X_2(0) = a_2)$. Subtracting (7.2.5) from (7.2.2), we obtain after differentiation

$$\frac{d}{dt}(X_1 - \overline{X_1}) = S(X_2 - a_2) + v_1. \tag{7.2.6}$$

The variance of $X_2 - a_2$ grows linearly at large times, but the variance of $v_1$ is constant. Hence, for large times the first term of (7.2.6) dominates and the second term may be neglected. Differentiating (7.2.6) once more, we obtain

$$\frac{d^2}{dt^2}(X_1 - \overline{X_1}) = S\frac{dX_2}{dt} = Sv_2. \tag{7.2.7}$$

This shows that $X_1 - \overline{X}_1$ is a double integral of a stationary function. According to (6.5.16), it asymptotically has a Gaussian distribution, whose variance is given by (Corrsin, 1953)

$$\overline{(X_1 - \overline{X}_1)^2} = \tfrac{2}{3} S^2 \, \overline{u_2^2} \, t^3 \, \mathcal{T}_{22}. \tag{7.2.8}$$

The dispersion in the $x_1$ direction thus increases much faster than the dispersion in the $x_2$ and $x_3$ directions. The latter are given by

$$\overline{(X_2 - a_2)^2} = 2 \, \overline{u_2^2} \, t \, \mathcal{T}_{22}, \quad \overline{(X_3 - a_3)^2} = 2 \, \overline{u_3^2} \, t \, \mathcal{T}_{33}. \tag{7.2.9}$$

In (7.2.8) and (7.2.9) the Lagrangian variance $\overline{v_\alpha^2}$ has been replaced by $\overline{u_\alpha^2}$ because the turbulence is homogeneous (7.1.8).

**Joint statistics**  If we want to predict the average shape of a patch of pollutant (smoke particles, say) released in a shear flow, the joint statistics of $X_1 - \overline{X}_1$ and $X_2 - a_2$ have to be analyzed. With a considerable amount of algebra, it can be shown that $X_1 - \overline{X}_1$ and $X_2 - a_2$ are jointly Gaussian at large times and that their covariance is given by

$$\overline{(X_1 - \overline{X}_1)(X_2 - a_2)} = \overline{u_2^2} St^2 \int_0^t \left(1 - \frac{\tau}{t}\right) \rho(\tau) \, d\tau = \overline{u_2^2} \, St^2 \mathcal{T}_{22}. \tag{7.2.10}$$

At large times, the correlation coefficient between $X_1 - \overline{X}_1$ and $X_2 - a_2$ is $\tfrac{1}{2}\sqrt{3}$; contours of constant probability density are given by

$$\frac{x^2}{\sigma_1^2} - \sqrt{3} \, \frac{xy}{\sigma_1 \sigma_2} + \frac{y^2}{\sigma_2^2} = \text{const.} \tag{7.2.11}$$

Here, $x = X_1 - \overline{X}_1$, $y = X_2 - a_2$; the variance $\sigma_1^2$ of $x$ is given by (7.2.8) and the variance $\sigma_2^2$ of $y$ is given by (7.2.9). The contours defined by (7.2.11) are ellipses; normalized with the standard deviation, as in (7.2.11), the ellipses have a constant aspect ratio, with a major axis of length $(1 + \tfrac{1}{2}\sqrt{3})^{1/2} \cong 1.37$ and a minor axis of length $(1 - \tfrac{1}{2}\sqrt{3})^{1/2} \cong 0.36$. The angle $\alpha$ between the major axis and the $x_1$ direction is given by

$$\tan \alpha = (\sigma_2 / \sigma_1)^{1/2} = \sqrt{3}/St. \tag{7.2.12}$$

As the patch moves downstream, the major axis rotates towards the horizontal (Figure 7.2). At large times, the patch becomes quite elongated.
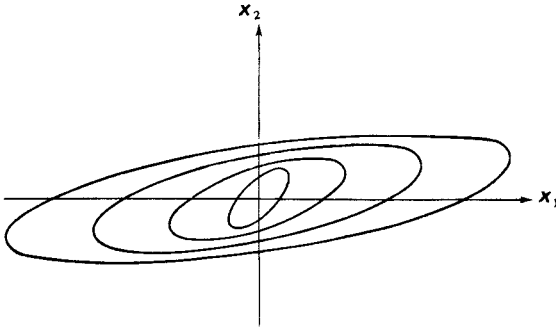
Figure 7.2. Dispersion in uniform shear flow. Equal increments of time are shown; all times are large compared to $\mathcal{T}_{22}$ (adapted from Corrsin, 1953).

**Longitudinal dispersion in channel flow**  Let us now consider dispersion in a channel flow. The mean flow is in the $x_1, x_2$ plane, $U_2 = U_3 = 0$, the height of the channel is $2h$. The position of a moving point is given by

$$X_1(\mathbf{a}, t) - a_1 = \int_0^t [U_1(X_2(\mathbf{a}, t')) + u_1(\mathbf{X}(\mathbf{a}, t'), t')] \, dt', \qquad (7.2.13)$$

$$X_2(\mathbf{a}, t) - a_2 = \int_0^t v_2(\mathbf{a}, t') \, dt', \qquad (7.2.14)$$

$$X_3(\mathbf{a}, t) - a_3 = \int_0^t v_3(\mathbf{a}, t') \, dt'. \qquad (7.2.15)$$

Here, $u_1(\mathbf{X}(\mathbf{a}, t'))$ is the Eulerian velocity fluctuation at the position of the moving point. For the reasons discussed in Section 7.1, $v_2$ and $v_3$ are stationary. Because $X_3 - a_3$ is the integral of a stationary function which itself is not a derivative of a stationary function, $X_3 - a_3$ asymptotically has a Gaussian distribution. Although $X_2 - a_2$ is also the integral of a stationary function, it does not have a Gaussian distribution because of the constraints imposed on $v_2$. Clearly, $X_2 - a_2$ itself is a stationary function because a moving point has to stay inside the channel. Therefore, $v_2$ is the derivative of a stationary function; its integral scale must be zero, and its spectrum behaves as $\omega^2$ near the origin, so that the central limit theorem does not apply (see Section 6.5).

The mean axial velocity of a moving point is given by (7.1.10); integrating this, we obtain

$$\overline{X}_1 = a_1 + U_b t. \qquad (7.2.16)$$

Here, as in (7.1.10), $U_b$ is the bulk velocity. Substitution of (7.2.16) into (7.2.13) yields

$$X_1 - \overline{X}_1 = \int_0^t [U_1(X_2) - U_b + u_1] \, dt'. \tag{7.2.17}$$

Because $X_2 - a_2$ is stationary, the moving point encounters $U_1 (X_2)$ and $u_1 (X(a,t'), t')$ in a stationary way: as far as the moving point is concerned, the integrand in (7.2.17) is stationary. The integrand is not a derivative, so that $X_1 - \overline{X}_1$ is Gaussian at large times. The variance of $X_1 - \overline{X}_1$ is then given by

$$\overline{(X_1 - \overline{X}_1)^2} = 2 v_1^2 t \mathscr{T}. \tag{7.2.18}$$

The Lagrangian velocity variance $\overline{v_1^2}$ was computed for pipe flow in (7.1.13); transposing this to channel flow, we have

$$\overline{v_1^2} = \frac{1}{2h} \int_{-h}^h [\overline{u_1^2} + (U_1 - U_b)^2] \, dx_2. \tag{7.2.19}$$

In the core region of channel flow at large Reynolds numbers, we may write (Section 5.2)

$$(U_1 - U_b)/u_* = F(\eta), \quad \overline{u_1^2}/u_*^2 = g(\eta). \tag{7.2.20}$$

Here, $\eta = x_2/h$. Substituting (7.2.20) into (7.2.19), we obtain

$$\overline{v_1^2} = u_*^2 \left( \int_0^1 F^2 \, d\eta + \int_0^1 g \, d\eta \right) = A u_*^2. \tag{7.2.21}$$

The constant $A$ is approximately equal to 5. In the wall layer, (7.2.20) is not valid; however, the wall layer is so thin that it makes a negligible contribution to the integrals. The integral scale $\mathscr{T}$ in (7.2.18) should be of order $h/u_*$, because $u_*$ and $h$ are the velocity and length scales of the core region of channel flow (Section 5.2). With this estimate and (7.2.21), (7.2.18) becomes

$$\overline{(X_1 - \overline{X}_1)^2} = C u_* h t, \tag{7.2 22}$$

where $C$ should be approximately equal to 10.

Channel flow is difficult to set up in a laboratory; however, experimental values for $C$ in pipe flow indeed range around 10 if the pipe radius instead of the channel half-width $h$ is used in the formula for the variance (Monin and Yaglom, 1971). Of course, (7.2.22) is valid only for $t \gg h/u_*$

**Bulk velocity measurements in pipes** Equations (7.2.16) and (7.2.22) may be applied to the problem of determining $U_b$ in pipes with tracer methods. If the pipe radius $D/2$ is used instead of the channel half-width $h$ in (7.2.22), the relative error in the measurement of bulk velocity is

$$\frac{\overline{[(X_1 - \overline{X}_1)^2]}^{1/2}}{\overline{X}_1 - a_1} = \left(\frac{Cu_* D}{2U_b^2 t}\right)^{1/2} = \left(\frac{u_*}{U_b}\right)^{1/2} \left(\frac{CD}{2U_b t}\right)^{1/2}. \tag{7.2.23}$$

The measurement is performed by releasing a patch of tracer material at $X_1 = a_1$ at time $t = 0$; the time interval $t$ between release and the passage of the patch at some downstream location is measured. The factor $U_b t/D$ is the streamwise distance in diameters; clearly, the accuracy of the measurement improves as this distance increases. The ratio $u_*/U_b$ is the square root of the friction coefficient; at typical Reynolds numbers, its value is about 0.04, so that $(u_*/U_b)^{1/2} \cong 0.2$. If $C = 10$, a streamwise separation of 100 diameters gives about a 4% standard deviation in the measurement of $U_b$. The accuracy can be improved considerably if the streamwise concentration distribution of the patch of tracer material is measured at the downstream location.

## 7.3
### Dispersion of contaminants
So far, we have discussed only the dispersion of moving points and assumed that it would be possible to mark or tag a Lagrangian "point" in such a way that it would keep its identity. In the two examples given in Section 7.2 we assumed without discussion that the motion of a minute tracer particle is identical to the motion of the Lagrangian point of fluid that would occupy the position of the particle if it were not there. Now, we have to consider more realistic dispersion problems. Two questions arise. First, contaminants are commonly released with some initial concentration distribution, so that the concentration distribution at later times has to be predicted. Second, contaminants are also dispersed by molecular transport, which may interact with the turbulent transport. We will discuss these problems separately.

**The concentration distribution** Let us consider contaminants which are not dispersed by molecular motion. This is an idealization; however, in liquids the molecular transport of contaminants (salinity, heat) is poor and in air the molecular transport of minute tracer particles (smoke, say) is poor, so that the assumption of zero diffusivity should be fairly realistic in those cases.

The transport of a contaminant with zero diffusivity is governed by

$$\frac{\partial \tilde{c}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{c}}{\partial x_j} = 0. \tag{7.3.1}$$

Here, $\tilde{c}$ is the instantaneous concentration at a point $x_j$, $t$, and $\tilde{u}_j$ is the instantaneous fluid velocity at that point. The solution of (7.3.1) is

$$\tilde{c}(\mathbf{X}(\mathbf{a}, t), t) = \tilde{c}(\mathbf{a}, 0). \tag{7.3.2}$$

This states that the concentration at each moving point remains equal to its value at the time of release. Because there is no molecular diffusion, this result is obvious. If we want to predict the mean concentration $C(\mathbf{x}, t)$, (7.3.2) has to be inverted. This is a backward dispersion problem: instead of asking where a point that started from $a_j$ at time $t = 0$ will go to, we are asking where a point that arrives at $x_j$ at time $t$ came from. In other words, we need a Lagrangian displacement integral like (7.1.1), but with time running backwards.

If the Lagrangian velocity field is stationary, the backward and forward dispersion problems are the same. If $B(\mathbf{X}, \mathbf{a}, t)$ is the probability density of $X_j(\mathbf{a}, t)$ for points that started at $a_j$ at time $t = 0$, then $B(\mathbf{a}, \mathbf{x}, t)$ is the probability density of the original positions $a_j(\mathbf{x}, t)$ of points that arrive at $x_j$ at time $t$. If $C(\mathbf{x}, t)$ is the mean concentration, we can write

$$C(\mathbf{x}, t) = \iiint\limits_{-\infty}^{\infty} \tilde{c}(\mathbf{a}, 0) B(\mathbf{a}, \mathbf{x}, t) \, da_1 \, da_2 \, da_3. \tag{7.3.3}$$

This states that the mean concentration at a point is the concentration carried by a particle times the probability of the particle being there, integrated over all particles that could be there.

If the initial concentration is all at one point $(a_j^0, \text{say})$, we have

$$\tilde{c}(\mathbf{a}, 0) = 0 \text{ for all } a_j \neq a_j^0, \tag{7.3.4}$$

$$\iiint\limits_{-\infty}^{\infty} \tilde{c}(\mathbf{a}, 0) \, da_1 \, da_2 \, da_3 = 1. \tag{7.3.5}$$

The integral (7.3.5) has been normalized for convenience. Equations (7.3.4) and (7.3.5) define a Dirac delta function $\delta(\mathbf{a} - \mathbf{a}^0)$; the integral (7.3.3) reduces to

$$C(\mathbf{x}, t) = \iiint\limits_{-\infty}^{\infty} \delta(\mathbf{a} - \mathbf{a}^0) B(\mathbf{a}, \mathbf{x}, t) \, da_1 \, da_2 \, da_3 = B(\mathbf{a}^0, \mathbf{x}, t). \tag{7.3.6}$$

The mean concentration is then equal to the probability density of the position of a moving point leaving from $a_i^0$. This suggests that $B(a^0, x)$ can be measured by introducing a small point source of contamination at $a_i^0$ and measuring the mean concentration throughout the field. The omission of the argument $t$ in $B$ is intentional: in practice, continuous point sources with constant flux are used, so that $C$ and $B$ are independent of time.

It is clear from (7.3.6) that minute tracer particles and nondiffusing contaminants indeed may be used to mark Lagrangian points. The conclusions obtained in Sections 7.1 and 7.2 thus apply to the concentration distribution as well as to the probability density; however, it must be kept in mind that the identification can be made only if the Lagrangian velocity field is stationary.

**The effects of molecular transport**  If the contaminant has a finite molecular diffusivity $\gamma$, the conservation equation for $\tilde{c}$ becomes

$$\frac{\partial \tilde{c}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{c}}{\partial x_j} = \gamma \frac{\partial^2 \tilde{c}}{\partial x_j \partial x_j}. \tag{7.3.7}$$

The presence of molecular diffusion makes it impossible to write (7.3.2), so that we have to proceed in a different way. The general problem raised by (7.3.7) is intractable; we consider the special case of a small spot of contaminant, centered around a moving point. Let us change to coordinates moving with the wandering point. If $\xi_i$ is the difference between the Eulerian position $x_i$ and the position of the Lagrangian point $X_i$, (7.3.7) becomes

$$\frac{\partial \tilde{c}}{\partial t} + \frac{\partial}{\partial \xi_j}[\tilde{c}(\tilde{u}_j(\xi) - \tilde{u}_j(0))] = \gamma \frac{\partial^2 \tilde{c}}{\partial \xi_j \partial \xi_j}. \tag{7.3.8}$$

Here, the continuity equation has been used to bring $\tilde{u}_j$ inside the derivative; of course, seen from a coordinate system moving with a Lagrangian point, the Eulerian velocity is not $\tilde{u}_j(x)$ but $\tilde{u}_j(\xi) - \tilde{u}_j(0)$. Equation (7.3.8) describes dispersion relative to a moving point. If the patch of contaminant is smaller than the Kolmogorov microscale, the velocity distribution in the neighborhood of the moving point is approximately linear:

$$\tilde{u}_j(\xi) - \tilde{u}_j(0) = \xi_j \frac{\partial \tilde{u}_j}{\partial \xi_j}(0). \tag{7.3.9}$$

The velocity field around the moving point is then a combination of a solid-body rotation (corresponding to the skew-symmetric part of $\partial \tilde{u}_j / \partial \xi_j$) and a

pure strain (corresponding to the symmetric part of $\partial \widetilde{u}_i/\partial \xi_j$). The value of $\partial \widetilde{u}_i/\partial \xi_j$ at $\xi_j = 0$, of course, generally varies in time.

Substitution of (7.3.9) into (7.3.8) yields

$$\frac{\partial \widetilde{c}}{\partial t} + \frac{\partial \widetilde{u}_i}{\partial \xi_j} \xi_j \frac{\partial \widetilde{c}}{\partial \xi_i} = \gamma \frac{\partial^2 \widetilde{c}}{\partial \xi_i \partial \xi_i} . \qquad (7.3.10)$$

It is easy to see that the total amount of contaminant in the spot must be conserved:

$$\int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} \widetilde{c} \, d\xi_1 \, d\xi_2 \, d\xi_3 = 1. \qquad (7.3.11)$$

The integral has been normalized for convenience. The shape and size of the spot can be measured by $I_{pq}$, which is defined by

$$\int\!\!\!\int\!\!\!\int_{-\infty}^{\infty} \xi_p \xi_q \widetilde{c} \, d\xi_1 \, d\xi_2 \, d\xi_3 = I_{pq} . \qquad (7.3.12)$$

The sum of the diagonal components of $I_{pq}$ is $I_{pp}$; this is proportional to the square of the average spot radius. The equation for $I_{pq}$ can be obtained from (7.3.10); it reads

$$\frac{d \, I_{pq}}{dt} - I_{pj} \frac{\partial \widetilde{u}_q}{\partial \xi_j} - I_{qj} \frac{\partial \widetilde{u}_p}{\partial \xi_j} = 2\gamma \delta_{pq}. \qquad (7.3.13)$$

If $\partial \widetilde{u}_i/\partial \xi_j$ is equal to zero, the solution of (7.3.13) is straightforward:

$$I_{pq} = 2 \gamma t \, \delta_{pq}. \qquad (7.3.14)$$

This states that, in the absence of relative motion near a point, the spot of contaminant is round ($I_{pq} = 0$ if $p$ and $q$ are different) and that it spreads by molecular diffusion in all directions. The radius of the spot is proportional to $I_{pp}^{1/2}$; clearly, the radius increases as $(\gamma t)^{1/2}$, as in all diffusion problems.

**The effect of pure, steady strain** Equation (7.3.13) cannot easily be solved for a general velocity field. However, the solution of a special case is instructive. Let us restrict the analysis to the effects of pure strain. Take a two-dimensional strain-rate field in which $\partial u_1/\partial \xi_1 = s$, $\partial u_2/\partial \xi_2 = -s$, $\partial u_3/\partial \xi_3 = 0$, and in which all off-diagonal components of $\partial \widetilde{u}_i/\partial \xi_j$ are zero. This represents pure, plane strain with stretching in the $\xi_1$ direction and compression in the $\xi_2$ direc-

tion. The approximation (7.3.9) implies that $s$ is uniform; we also assume that it does not vary in time. The choice of the symbol $s$ is not an arbitrary one. We found in Chapter 3 that the strain-rate fluctuations $s_{ij}$ in turbulence are quite large. We recall that $s_{ij} \sim u/\lambda$ ($\lambda$ is the Taylor microscale); these large strain rates are associated with the small-scale motion. For the small spot we are considering here, we may thus consider $s$ to be of order $u/\lambda$.

For steady, plane strain, (7.3.13) becomes

$$dI_{11}/dt - 2sI_{11} = 2\gamma, \tag{7.3.15}$$

$$dI_{22}/dt + 2sI_{22} = 2\gamma, \tag{7.3.16}$$

$$dI_{12}/dt = 0. \tag{7.3.17}$$

The solution of (7.3.15–7.3.17) is

$$I_{11} = 2\gamma \frac{\exp(2st) - 1}{2s}, \qquad I_{22} = 2\gamma \frac{1 - \exp(-2st)}{2s}, \tag{7.3.18}$$

$$I_{11} + I_{22} = 4\gamma \frac{\sinh(2st)}{2s}, \tag{7.3.19}$$

$$I_{12} = 0. \tag{7.3.20}$$

For very small total strain $st$, $\sinh(2st) \cong 2st$, so that $I_{11} + I_{22} \cong 4\gamma t$, which agrees with (7.3.14). However, as the strain $st$ increases, (7.3.19) increases much faster than $t$, so that the spot spreads much faster than it would as a result of molecular transport alone. The straining motion thus accelerates molecular diffusion of small spots. In turbulence this effect is quite pronounced, because the fluctuating strain rates are so large.

The cause of the accelerated diffusion is easy to understand. As a spot of contaminant is drawn out in the $\xi_1$ direction (Figure 7.3), the concentration gradients in that direction are reduced. Because the diffusion of contaminant is proportional to the concentration gradient, the rate of spread in the $\xi_1$ direction is reduced. In the $\xi_2$ direction, however, the spot is being compressed, so that the gradients and the molecular diffusion in the $\xi_2$ direction increase. At small values of $st$, the increase in the $\xi_2$ gradient is about equal to the decrease in the $\xi_1$ gradient, but at large values of $st$ the increase of diffusion in the $\xi_2$ direction is much larger than the decrease of diffusion in the $\xi_1$ direction, so that the net rate of diffusion increases as indicated by (7.3.19).

The interaction of turbulent and molecular transport thus results in much
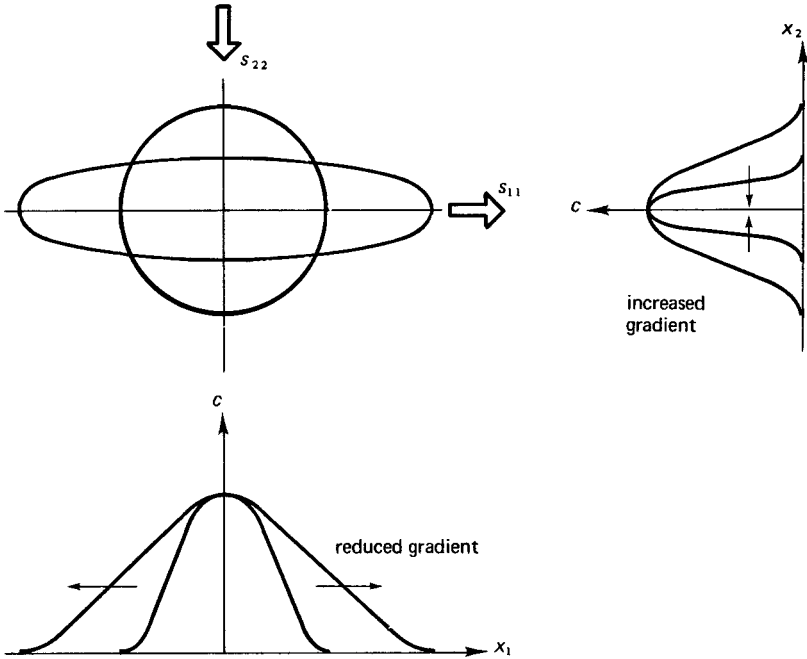
Figure 7.3.   Effect of strain on concentration gradients.

faster spreading of the spot. This is one of the reasons why turbulent mixing is so effective. If there were no molecular transport, turbulent mixing would carry thin sheets and filaments of contaminant to every part of the flow. However, there would still be large inhomogeneities at small scales, because the filaments would be separated by regions of uncontaminated fluid, which would have to be filled by unaccelerated molecular diffusion.

The expressions (7.3.18) provide support for the calculations of the minimum scale in cases with $\gamma/\nu < 1$, given in Chapter 3 (see (3.3.68), (3.4.7), and Figure 3.6). Examining the expression for $l_{22}$, we see that it never gets smaller than $\gamma/s$, no matter how large the total strain becomes. The minimum scale then is $(\gamma/s)^{1/2}$. On substitution of $s$ by $u/\lambda \sim (\epsilon/\nu)^{1/2}$, the contaminant microscale becomes $(\gamma/\nu)^{1/2}\eta$.

The assumption that the strain-rate field is steady is not unrealistic. As we saw in Chapter 3, time derivatives of the vorticity and strain-rate fields are of order $R_\ell^{-1/2}$ relative to $1/s$. In other words, the straining goes on for many times $1/s$. Of course, the strain rate eventually changes sign, so that the rate

of spreading is controlled not by $st$, but by $s(t\mathcal{T})^{1/2}$, where $\mathcal{T}$ is the Lagrangian integral scale of the strain-rate field. The assumption that the strain-rate field is infinite compared to the spot size clearly corresponds to $\gamma/\nu \ll 1$. The assumption that the vorticity is zero is not vital to the argument; a more exact calculation that includes the vorticity does not change the conclusions obtained here (Lumley, 1972 b).

**Transport at large scales**  The effects of turbulence-accelerated molecular transport are mainly confined to small scales because the strain-rate fluctuations are most intense at small scales. As we have seen above, molecular diffusion rapidly removes the small-scale concentration inhomogeneities created by the straining motion. This interaction tends to make the concentration distribution approximately homogeneous at small scales. The time needed for homogenizing may be large compared to $(\nu/\epsilon)^{1/2}$, but $(\nu/\epsilon)^{1/2} \sim R_\ell^{-1/2} \, \ell/u$ (1.5.15), so that this time scale is likely to be small compared to the large-eddy time scale $\ell/u$.

If the instantaneous concentration $\tilde{c}$ is decomposed into a mean concentration $C$ and concentration fluctuations $c$, the conservation equation for $C$ becomes (in the absence of mean flow)

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x_i} \overline{cu_i} = \gamma \frac{\partial^2 C}{\partial x_i \partial x_i} \, . \tag{7.3.21}$$

The transport term on the left-hand side of (7.3.21) is of order $Cu\ell_c$ ($\ell_c$ is a length scale characteristic of mean concentration gradients). The molecular diffusion term is of order $\gamma C/\ell_c^2$. The ratio of these is $u\ell_c/\gamma$. Because turbulence-accelerated diffusion increases $\ell_c$ rapidly, $u\ell_c/\gamma$ (which is comparable to the Reynolds number if $\gamma/\nu \cong 1$, as in gases) tends to become large, so that the effects of molecular diffusion on the mean concentration distribution can often be neglected. This conclusion, of course, is identical to the one obtained for the transport of mean momentum (Section 2.1).

## 7.4
### Turbulent transport in evolving flows
In the preceding sections we have discussed only cases in which the Lagrangian velocities were stationary. The problem becomes much more difficult if they are not. Nonstationary Lagrangian velocities arise if the Eulerian flow field is nonstationary or inhomogeneous (or both); in this section, we discuss

the dispersion of contaminants in self-preserving, inhomogeneous, statistically steady flows.

**Thermal wake in grid turbulence**   Consider transport of heat released from a line source in grid turbulence. The turbulence is produced in a wind tunnel. The mean velocity $U$ is in the $x$ direction; it is assumed to be uniform. The turbulence is homogeneous in the $y, z$ plane, but it decays downstream. The (Eulerian) integral scale increases as $x^{1/2}$ downstream, while the turbulent energy $\frac{1}{2} u^2$ decreases as $x^{-1}$, to a first approximation (Section 3.2).

The line source could be a heated wire stretched across the wind tunnel; we assume that the heat supply is steady. The wire produces a small temperature rise in all the material points that happen to pass through its boundary layer. The heated wake of the wire is slowly broadened by the turbulence-accelerated molecular transport, but it is also carried from side to side by larger eddies (Figure 7.4). If the mean temperature difference between any point within the thermal wake and the unheated fluid is called $\Theta$ and if the temperature fluctuations are designated by $\theta$, the equation for $\Theta$ reads

$$U \frac{\partial \Theta}{\partial x} + \frac{\partial}{\partial x} (\overline{u\theta}) + \frac{\partial}{\partial y} (\overline{v\theta}) = \gamma \frac{\partial^2 \Theta}{\partial x^2} + \gamma \frac{\partial^2 \Theta}{\partial y^2}. \tag{7.4.1}$$

The second and fourth terms of (7.4.1) are small, as can easily be demonstrated by repeating the order-of-magnitude analysis for plane wakes (Section 4.1). The last term of (7.4.1) is also small, but we will retain it to see what effect molecular transport has on the distribution of $\Theta$. Consequently, (7.4.1) is approximated by

$$U \frac{\partial \Theta}{\partial x} + \frac{\partial}{\partial y} (\overline{v\theta}) = \gamma \frac{\partial^2 \Theta}{\partial y^2}. \tag{7.4.2}$$
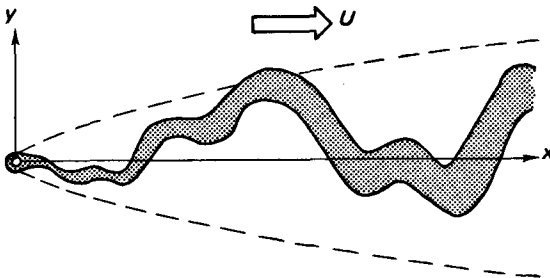


Figure 7.4.   Definition sketch for plane thermal wake.

Integrating (7.4.2), we obtain

$$U \int_{-\infty}^{\infty} \Theta \, dy = \frac{H}{\rho c_p}. \tag{7.4.3}$$

The total flux of heat past any downstream location is thus constant. This relation is similar to the momentum integral in ordinary wakes.

**Self-preservation** We are looking for a self-preserving solution to (7.4.2, 7.4.3). Immediately, a problem arises. The turbulence has a length scale $\ell$, whose growth is fixed; if the temperature distribution has another length scale, which might increase at a different rate, self-preservation cannot exist. If the virtual origin of the thermal wake is the same as the virtual origin of the turbulence, this problem would not arise. This could be arranged by putting the heated wire very close to the grid or, even better, by heating one of the bars of the grid. If the heated wire is at some distance from the grid, however, self-preservation does not seem possible. If the mean temperature difference at the center line of the wake is called $\Theta_0$ and if the length scale of the thermal wake is $\ell_\theta$, the turbulent transport term in (7.4.2) is of order

$$-\frac{\partial}{\partial y} (\overline{v\theta}) = \mathcal{O} \left( u \Theta_0 / \ell_\theta \right). \tag{7.4.4}$$

If the thermal wake is self-preserving, because the heated wire is located near the grid, the transport term is

$$-\frac{\partial}{\partial y} (\overline{v\theta}) = \mathcal{O} \left( u \Theta_0 / \ell \right). \tag{7.4.5}$$

The values of $\Theta_0$ in (7.4.4) and (7.4.5) are not the same; the heat flux is the same in both cases, the value of $\Theta_0$ at some given downstream distance $x$ from the grid is larger for the wake of the wire that is closest to $x$. Also, close behind that wire $\ell_\theta \ll \ell$, so that (7.4.4) produces abnormally large turbulent transport in the $y$ direction. This causes rapid broadening of the temperature distribution, so that we may expect $\ell_\theta$ to catch up with $\ell$ (Figure 7.5).

Another way to understand this effect is to take account of the fact that the width of the distribution $\Theta$ increases roughly proportionally to the square root of the time since release for all but very small times. At a given mean velocity $U$, the width thus increases as the square root of the distance from the wire; if the distance from the wire is much smaller than the distance from
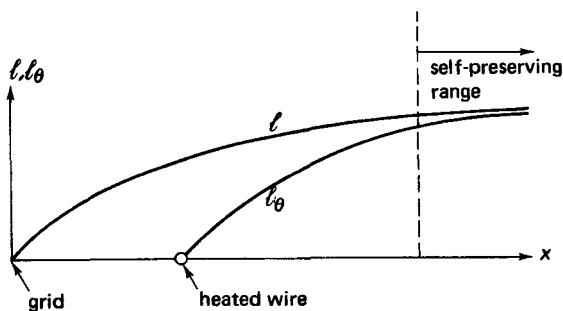
Figure 7.5. The growth of $\ell_\theta$ for a source not located near the virtual origin of the turbulence.

the turbulence-producing grid, $\ell_\theta$ increases faster than $\ell$. Therefore, if we allow some distance for $\ell_\theta$ to become comparable to $\ell$, a self-preserving solution should be feasible.

The assumption of self-preservation consists of

$$\Theta = \Theta_0\, f(y/\ell), \quad -\overline{\theta v} = \Theta_0\, u g(y/\ell). \tag{7.4.6}$$

Here, $\Theta_0 = \Theta_0(x)$ and $\ell = \ell(x)$. We have assumed that $\ell_\theta = \ell$; of course, the self-preserving decay of the turbulence prescribes $u \propto x^{-1/2}$, $\ell \propto x^{1/2}$. Substitution of (7.4.6) into (7.4.2) yields

$$\frac{U}{u}\frac{\ell}{\Theta_0}\frac{d\Theta_0}{dx}f - \frac{U}{u}\frac{d\ell}{dx}f'\eta = g' + \frac{\gamma}{u\ell}f''. \tag{7.4.7}$$

Here, primes denote differentiation with respect to $\eta(=y/\ell)$. Self-preservation can be obtained only if the coefficients in (7.4.7) are constant:

$$\frac{U}{u}\frac{\ell}{\Theta_0}\frac{d\Theta_0}{dx} = A, \quad \frac{U}{u}\frac{d\ell}{dx} = B, \quad \frac{\gamma}{u\ell} = \frac{1}{P}. \tag{7.4.8}$$

Because $u \propto x^{-1/2}$ and $\ell \propto x^{1/2}$, the second and third of (7.4.8) are satisfied ($P$ is a Péclet number). The first of (7.4.8) can be satisfied by any power law $\Theta_0 \propto x^n$, but the heat flux integral (7.4.3) requires that $\Theta_0\ell$ be constant, so that $\Theta_0$ varies as $x^{-1/2}$. This is not surprising, because $\Theta_0$ is similar to the center-line velocity difference $U_s$ in momentum wakes (Section 4.1). It is convenient that the molecular transport term is also self-preserving; it will be retained. With these results, (7.4.7) becomes

$$-B(f + \eta f') = g' + P^{-1} f''. \tag{7.4.9}$$

Integration of (7.4.9) yields

$$-B\eta f = g + P^{-1} f'. \tag{7.4.10}$$

Let us assume that the eddy diffusivity is constant. This is a much better assumption than in the wakes studied in Section 4.1, because this flow has no edges, is not intermittent, and is homogeneous in the cross-stream direction. If the eddy diffusivity is $\gamma_T$ and the nondimensional group $u\ell/\gamma_T$ is called the turbulent Péclet number $P_T$, (7.4.10) becomes

$$-B\eta f = (P_T^{-1} + P^{-1}) f'. \tag{7.4.11}$$

It is convenient to define $\ell$ by

$$B = \frac{U}{u} \frac{d\ell}{dx} \equiv P_T^{-1} + P^{-1}. \tag{7.4.12}$$

The solution of (7.4.11) then becomes

$$f = \exp\left(-\tfrac{1}{2}\eta^2\right). \tag{7.4.13}$$

The mean temperature difference $\Theta$ thus has a Gaussian distribution, just like the momentum deficit in wakes (4.2.15).

It is clear from (7.4.11, 7.4.12) that, to the degree of approximation used here, the effect of molecular transport on the mean temperature distribution is additive. If $P_T$ is of the same order as $R_T$ in plane wakes ($R_T = 12.5$, see Table 4.1) and if $P$ is at all large, the additional spreading due to molecular transport is negligible.

**Dispersion relative to the decaying turbulence**   It has been assumed that the width of the temperature wake scales with the length of the decaying turbulence, which increases as $x^{1/2}$. This implies that the dispersion, nondimensionalized with the local length scale, does not increase as soon as self-preservation has been attained. Clearly, wandering points are not being dispersed in the sense used earlier in this chapter. This peculiar behavior arises because the grid turbulence "disperses" its own length scales at a rate consistent with the dispersion of contaminants; it is characteristic of dispersion in evolving flows such as jets, wakes, and boundary layers.

If the heated wire is not located close to the grid, self-preservation is unlikely to be observed experimentally. The time scale $\ell/u$ of the turbulence

is of the same order as $t = x/U$ (3.2.32). It takes the length scale of the temperature wake several $\ell/u$ to catch up with the length scale of the turbulence. However, $\ell/u$ is also the time scale of decay, so that after several $\ell/u$ the turbulence is no longer self-preserving, but has entered the final period of decay, in which $\ell$ and $u$ change downstream in a different way.

**The Gaussian distribution**  The result that the distribution of $\Theta$ is Gaussian (within the assumption of constant eddy diffusivity) is in good agreement with experimental data, even at small distances from the point of release. This should not be construed as support for a constant eddy diffusivity, because the probability density of the velocity fluctuations is also observed experimentally to be approximately Gaussian at all times, so that the Gaussian distribution of $\Theta$ would seem to be an unavoidable result. In fact, the position of a wandering point, nondimensionalized with the local length scale $\ell$, itself becomes a stationary variable at a large distance from the grid; there is no reason why it should have a Gaussian distribution, except for the dynamics of turbulence which happen to make it so.

**Dispersion in shear flows**  The analysis presented in this section may also be applied to dispersion by other self-preserving flows, such as jets, wakes, plumes, and boundary layers. Some time after release, the plume of contaminant will have spread throughout the turbulent part of the flow; beyond that, dispersion of momentum and dispersion of contaminant go hand in hand, just as in the thermal plume discussed in Section 4.6. Because the contaminant cannot spread beyond the edges of the flow, the length scale of the contaminant distribution remains the same as the length scale of the flow.

If the point of release of contaminant does not coincide with the virtual origin of the flow, we cannot expect self-preservation near the point of release. Because shear flows exhibit no cross-stream homogeneity, the initial dispersion problem is extremely complicated. Sometimes, approximate solutions are obtained by assuming that the turbulence is homogeneous and that the mean velocity $U$ is approximately constant in the neighborhood of the point of release; the initial dispersion can then be described with the analysis of Section 7.1, where the time $t$ since release is replaced by $x/U$. The effect of mean shear is sometimes accounted for by assuming that the mean velocity gradient is approximately constant; the results obtained in Section 7.2 may then lead to qualitatively correct conclusions.

## Problems

**7.1**  A chemical is added at the center line of a fully developed turbulent pipe flow. The reaction rate is large, so the total reaction time is determined by turbulent transport. How many diameters are required for the reaction to be completed?

**7.2**  A kilogram of a half-and-half mixture of two fluids is being homogenized by a 25-watt mixer. The two fluids have about the same viscosity and density (about $10^{-5}$ $m^2$/sec and 1 $kg/m^3$, respectively); the diffusivity of one fluid into the other is about $3 \times 10^{-6}$ times the viscosity. This situation occurs if one of the fluids is a dilute solution of high molecular weight polymers. Make a conservative estimate of the mixing time required for homogeneity of the mixture. Suppose that in the mixing process it is necessary to use only strain rates small compared to $5 \times 10^2$ $sec^{-1}$, because larger strain rates tend to tear the polymer molecules apart. If you limit yourself to strain rates one-tenth of this value, what mixing time is required? What is the power of the mixer in this case? If the mixer paddle is 5 cm in diameter, is the flow turbulent?

**7.3**  A smokestack located in the lower part of the atmospheric boundary layer releases a steady stream of neutrally buoyant smoke. Estimate the downstream position of the point of maximum pollutant concentration at the surface. What is the effect of the stack height on the maximum surface concentration?