4

BOUNDARY-FREE SHEAR FLOWS

Turbulent shear flows that occur in nature and in engineering are usually evolving that is, in the flow direction the structure of the flow is changing. This change is sometimes due to external influences, such as pressure or temperature gradients, and sometimes due only to evolutionary influences inherent within the turbulence. At the present time, very few evolving flows are well understood; those evolving because of external influences are particularly difficult to understand, unless the variation of the external influence happens to match in some way the flow's own evolutionary tendencies. In Section 5.5 we encounter an example of such a flow. Here we shall limit ourselves to flows evolving under the influence of their own evolutionary tendencies. Even this class of flows is not generally understood; we shall further restrict the discussion to two-dimensional flows whose evolution is slow and whose dynamics is not affected by the presence of a solid surface.

4.1
Almost parallel, two-dimensional flows

There are two types of two-dimensional flows, the so-called plane flows and the axisymmetric flows. In both, the mean velocity field is entirely confined to planes. In the plane flows, mean flow in planes parallel to a given plane is identical; in the axisymmetric flows, mean flow in planes through the axis of symmetry is identical. We analyze in detail the plane flows (for algebraic simplicity) and give the results for the axisymmetric flows.

Plane flows Let us consider flows whose principal mean-velocity component is in the x direction, which are confined to the x,y plane, and which evolve slowly in the x direction. Thus,

$$U_i = \{U, V, 0\}, \quad \partial/\partial x \ll \partial/\partial y \text{ nearly everywhere.} \qquad (4.1.1)$$

The classical flows falling within this class are wakes, jets, and shear layers (Figure 4.1). For these flows it is possible to simplify the equations of motion by discarding many terms that are small. To identify these terms, we must determine in what order the terms vanish as these flows become more and more nearly parallel. Slightly more complicated flows, such as jets flowing into a moving medium, are not treated here; they can be analyzed in the same way as the flows in Figure 4.1.

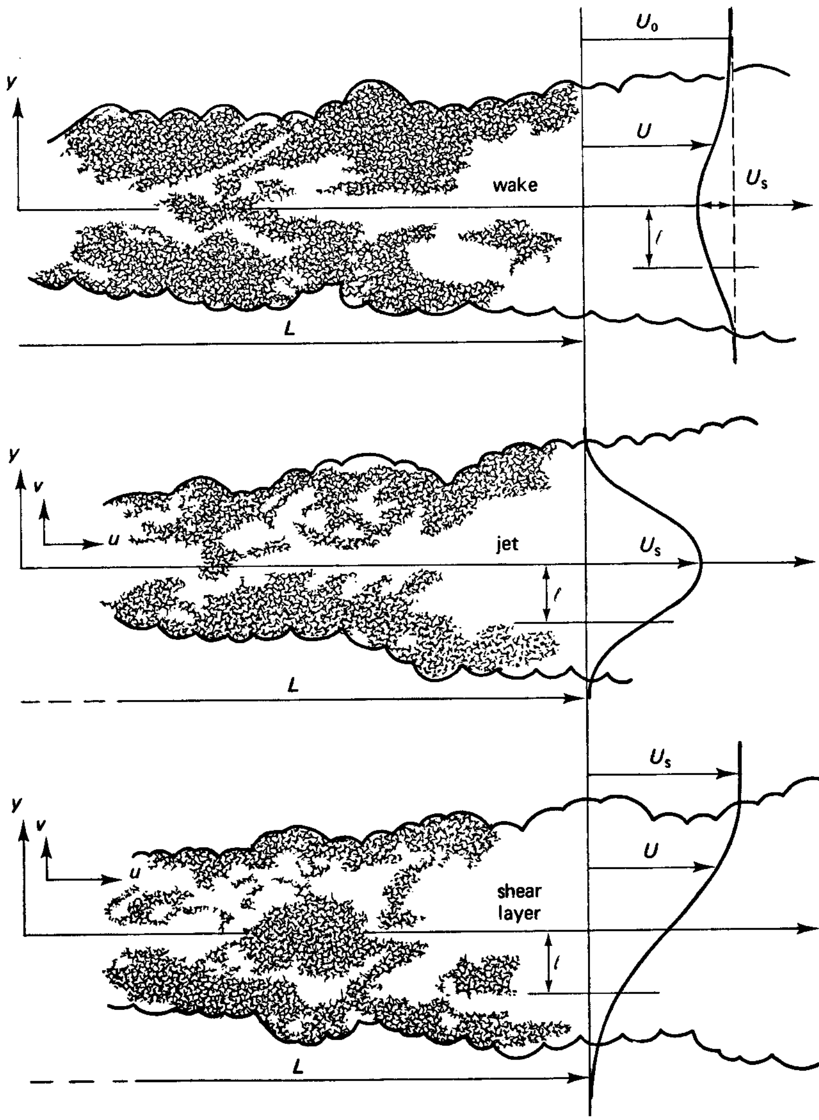

Figure 4.1. Plane turbulent wakes, jets, and shear layers (mixing layers).

Examining Figure 4.1, we can identify two velocity scales in the wake and one in the jet and the shear layer. In the wake, there is a scale U_0 for the velocity of the mean flow in the x direction; in all of these flows there is a scale U_s for the cross-stream variation of the mean velocity component in the x direction. Let us define U_s as the maximum value of $|U_0\text{-}U|$. In wakes $U_s \ll U_0$ far from the obstacle, while in jets and shear layers $U_0 = 0$. Hence, in far wakes $U = U_0 + (U - U_0) = \mathcal{O}(U_0 + U_s) = \mathcal{O}(U_0)$, while in jets and shear layers $U = \mathcal{O}(U_s)$ (as before, $\mathcal{O}$ stands for "order of magnitude"). For convenience we use $U = \mathcal{O}(\tilde{U})$, where $\tilde{U} = U_0$ for wakes and $\tilde{U} = U_s$ for jets and shear layers.

If we agree to define a cross-stream scale ℓ as the distance from the center line at which $U - U_0$ is about $\frac{1}{2}U_s$ (a more precise selection is made later), we can write

$$\partial U/\partial y = \mathcal{O}(U_s/\ell). \tag{4.1.2}$$

We designate the scale of change in the x direction by L, so that

$$\partial U/\partial x = \mathcal{O}(U_s/L). \tag{4.1.3}$$

In addition to the velocity and length scales just defined, we need a velocity scale for the turbulence. Let us use the symbol u, so that

$$-\overline{uv} = \mathcal{O}(u^2), \quad \overline{u^2} = \mathcal{O}(u^2), \quad \overline{v^2} = \mathcal{O}(u^2). \tag{4.1.4}$$

The magnitude of u relative to U_s is determined later. Finally, we need a scale for the cross-stream component V of the mean velocity. This scale can be determined from the mean equation of continuity:

$$\partial U/\partial x + \partial V/\partial y = 0. \tag{4.1.5}$$

Because $\partial U/\partial x \sim U_s/L$, we need $\partial V/\partial y \sim U_s/L$ in order to balance (4.1.5). On the other hand, cross-stream length scales are proportional to ℓ, so that $\partial V/\partial y \sim V/\ell$. Equating these two estimates, we obtain

$$V = \mathcal{O}(U_s \ell/L). \tag{4.1.6}$$

The cross-stream momentum equation We are now in a position to examine the equations of motion in the limit as $\ell/L \to 0$, that is, as the flow becomes parallel. Let us first look at the equation for V, which governs the mean momentum in the cross-stream direction. This equation is

$$U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+\frac{\partial}{\partial x}(\overline{uv})+\frac{\partial}{\partial y}(\overline{v^2})=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right). \tag{4.1.7}$$

Expressing each term of (4.1.7) in the scales introduced earlier, we may identify their orders of magnitude as follows:

$$\begin{aligned}
U\frac{\partial V}{\partial x}&: \quad \frac{\tilde{U}U_s\ell}{L\;L}=\left[\frac{\tilde{U}}{u}\frac{U_s}{u}\left(\frac{\ell}{L}\right)^2\right]\frac{u^2}{\ell},\\
V\frac{\partial V}{\partial x}&: \quad \left(\frac{U_s\ell}{L}\right)^2\frac{1}{\ell}=\left[\left(\frac{U_s}{u}\right)^2\left(\frac{\ell}{L}\right)^2\right]\frac{u^2}{\ell},\\
\frac{\partial}{\partial x}(\overline{uv})&: \quad \frac{u^2}{L}=\frac{\ell}{L}\cdot\frac{u^2}{\ell},\\
\frac{\partial}{\partial y}(\overline{v^2})&: \quad \frac{u^2}{\ell}=1\cdot\frac{u^2}{\ell},\\
\frac{1}{\rho}\frac{\partial P}{\partial y}&: \quad ?,\\
\nu\frac{\partial^2 V}{\partial x^2}&: \quad \frac{\nu U_s\ell}{L}\frac{1}{L^2}=\left[\frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right)^3\right]\frac{u^2}{\ell},\\
\nu\frac{\partial^2 V}{\partial y^2}&: \quad \frac{\nu U_s\ell}{L}\frac{1}{\ell^2}=\left[\frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right)\right]\frac{u^2}{\ell}.
\end{aligned} \tag{4.1.8}$$

Unless $u^2/(\tilde{U}U_s) \to 0$ as fast as $(\ell/L)^2$, the first, second, and third terms of (4.1.7) are negligible relative to $\partial\overline{v^2}/\partial y$. If the Reynolds number $R_\ell = u\ell/\nu$ is large enough, the viscous terms are also negligible compared to $\partial\overline{v^2}/\partial y$. There must be at least one term of the same order as $\partial\overline{v^2}/\partial y$ in order to balance the equation; inspection of (4.1.8) shows that only the pressure term can do this. Thus we obtain the following approximate form of (4.1.7):

$$\partial\overline{v^2}/\partial y = -(1/\rho)\,\partial P/\partial y. \tag{4.1.9}$$

This approximation is valid only if

$$\frac{\tilde{U}}{u}\frac{U_s}{u}\left(\frac{\ell}{L}\right)^2 \to 0, \quad \frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right) \to 0 \tag{4.1.10}$$

in the limit as $\ell/L \to 0$; the conditions (4.1.10) need to be imposed to assure the negligibility of the first two and the last two terms of (4.1.7). We shall later show that (4.1.10) is always satisfied, provided that R_ℓ is sufficiently large.

Integration of (4.1.9) is straightforward; it yields

$$P/\rho + \overline{v^2} = P_0/\rho. \tag{4.1.11}$$

Here, P_0 is the pressure outside the turbulent part of the flow field ($y \to \pm\infty$). Equation (4.1.11) holds for all narrow, slowly evolving flows. We will assume that the imposed downstream pressure gradient $\partial P_0/\partial x = 0$. If P_0 were to vary in the x direction we could not state without hesitation that all derivatives in the downstream direction scale with L, since the variation of P_0 might introduce another scale.

We need the derivative of (4.1.11) with respect to x. Because $\partial P_0/\partial x = 0$, we obtain

$$(1/\rho)\,\partial P/\partial x + \partial \overline{v^2}/\partial x = 0. \tag{4.1.12}$$

The streamwise momentum equation The equation for U, which governs the downstream component of mean momentum, reads

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) + \frac{\partial}{\partial y}(\overline{uv}) = \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right). \tag{4.1.13}$$

Here, (4.1.12) has been used to substitute for $\partial P/\partial x$. Using the scales already introduced, we estimate the orders of magnitude of the terms of (4.1.13) as

$$\begin{aligned}
U\frac{\partial U}{\partial x} &: \quad \tilde{U}\frac{U_s}{L} = \left[\frac{\tilde{U}}{u}\frac{U_s}{u}\frac{\ell}{L}\right]\frac{u^2}{\ell},\\
V\frac{\partial U}{\partial y} &: \quad \frac{U_s \ell}{L}\frac{U_s}{\ell} = \left[\left(\frac{U_s}{u}\right)^2\frac{\ell}{L}\right]\frac{u^2}{\ell},\\
\frac{\partial}{\partial x}(\overline{u^2} - \overline{v^2}) &: \quad \frac{u^2}{L} = \frac{\ell}{L}\cdot\frac{u^2}{\ell},\\
\frac{\partial}{\partial y}(\overline{uv}) &: \quad \frac{u^2}{\ell} = 1\cdot\frac{u^2}{\ell},\\
\nu\frac{\partial^2 U}{\partial x^2} &: \quad \frac{\nu U_s}{L^2} = \left[\frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right)^2\right]\frac{u^2}{\ell},\\
\nu\frac{\partial^2 U}{\partial y^2} &: \quad \frac{\nu U_s}{\ell^2} = \left[\frac{U_s}{u}\frac{1}{R_\ell}\right]\frac{u^2}{\ell}.
\end{aligned} \tag{4.1.14}$$

If we assume that R_ℓ is sufficiently large, we can make the viscous terms as small as desired. In the limit as $\ell/L \to 0$, the third term of (4.1.13) is also negligible. In order to balance the equation, at least one other term of the same order as $\partial(\overline{uv})/\partial y$ is needed. Of the remaining terms, the first is the largest because $\tilde{U} \gg U_s$. Thus, we must require that

$$\frac{\tilde{U}}{\mathscr{u}}\frac{U_s}{\mathscr{u}}\frac{\ell}{L} = \mathcal{O}(1); \tag{4.1.15}$$

that is, this nondimensional group must remain bounded as $\ell/L \to 0$.

Turbulent wakes There are two ways in which (4.1.15) can be satisfied. If, as one possible choice, we take $\mathscr{u}/U_s = \mathcal{O}(1)$, (4.1.15) requires that

$$\mathscr{u}/\tilde{U} = \mathcal{O}(\ell/L). \tag{4.1.16}$$

This situation occurs in far wakes. Far wakes have turbulence intensities of the order of the velocity defect; both of these are small relative to the mean velocity. As a wake evolves downstream and as ℓ/L becomes smaller, $\mathscr{u}/U$ keeps pace with it.

With $\mathscr{u} = \mathcal{O}(U_s)$ and (4.1.16), the second term in (4.1.13, 4.1.14) is negligible relative to the first, so that the momentum equation for turbulent wakes far from an obstacle reduces to

$$U\,\partial U/\partial x + \partial(\overline{uv})/\partial y = 0. \tag{4.1.17}$$

We can make one further simplification. For wakes, $\tilde{U} = U_0$ and $\mathscr{u} \sim U_s$, so that we may write, by virtue of (4.1.16),

$$(U - U_0)/U_0 = \mathcal{O}(U_s/U_0) = \mathcal{O}(\ell/L). \tag{4.1.18}$$

This implies that the undifferentiated U occurring in (4.1.17) may be replaced by U_0. Thus, (4.1.17) may be approximated by

$$U_0\,\partial U/\partial x + \partial(\overline{uv})/\partial y = 0. \tag{4.1.19}$$

This equation states that the net momentum flux due to the cross-stream velocity fluctuations v is replaced by x momentum carried by the mean flow in the streamwise direction.

Returning to the provisions expressed in (4.1.10), we see that the first is satisfied if $\ell/L \to 0$ and if (4.1.16) holds. The second provision is satisfied as long as $\ell/(R_\ell L) \to 0$. This condition can be met easily. If we examine (4.1.14), we see that the condition for neglecting the viscous terms in (4.1.13) is that $1/R_\ell \to 0$, which is more stringent than the second provision in (4.1.10). If the viscous terms are to be of the same order as the other terms that have been neglected, we must require the even stronger condition $1/R_\ell = \mathcal{O}(\ell/L)$. Hence, roughly speaking, (4.1.19) is a valid approximate equation of motion for far wakes provided $1/R_\ell \sim \ell/L \ll 1$.

Turbulent jets and mixing layers The second way in which (4.1.15) may be satisfied is by putting $\tilde{U} = U_s$, so that (4.1.15) becomes

$$u/U_s = \mathcal{O}(\ell/L)^{1/2}. \tag{4.1.20}$$

The choice describes jets and mixing layers, in which turbulence intensities are about half an order of magnitude (measured in terms of ℓ/L) smaller than the jet velocity or the velocity difference in the mixing layer (shear layer). With the choice (4.1.20) the first and second terms in (4.1.14) are of the same order, so that the appropriate momentum equation is

$$U\,\partial U/\partial x + V\,\partial U/\partial y + \partial(\overline{uv})/\partial y = 0. \tag{4.1.21}$$

Here, the x momentum removed by the cross-stream velocity fluctuations v is replaced by mean-flow convection carried by both the downstream and the cross-stream components of the mean velocity.

The provisions (4.1.10) need to be examined. We find that the first of these is satisfied if $\ell/L \to 0$ and if (4.1.20) holds. The second provision amounts to $(\ell/L)^{1/2}\ R_\ell^{-1} \to 0$. This appears to be an easy condition. From (4.1.14) we conclude that the condition for the negligibility of the major viscous term is that $(L/\ell)^{1/2}\ R_\ell^{-1} \to 0$, which is a fairly strong requirement. To assure that the viscous term is of the same order as the other terms which have been neglected, we need the even stronger condition $R_\ell = \mathcal{O}(L/\ell)^{3/2}$. We conclude that (4.1.21) is a correct approximation if $\ell/L \to 0$ and if $(L/\ell)^{1/2}\ R_\ell^{-1} \to 0$.

We shall find later that in wakes ℓ/L continually decreases downstream, so that (4.1.19) becomes a better approximation the farther downstream one goes. For mixing layers and jets, on the other hand, we shall find that ℓ/L is constant. The observed values of ℓ/L in jets and mixing layers are of the order 6×10^{-2}, so that the neglected terms in (4.1.21) amount to about 6% of the terms retained. In the various plane and axisymmetric wakes, jets, and shear layers we shall study the Reynolds number R_ℓ changes downstream in different ways. Hence, in each flow there are distinct regions in which the conditions on R_ℓ are satisfied.

The momentum integral Because (4.1.19) is a special case of (4.1.21), all relations based on (4.1.21) also hold for (4.1.19), so that we can confine further analysis to (4.1.21). If we subtract U_0 from U when the latter appears within the streamwise derivatives of (4.1.21), we obtain

$$U \frac{\partial}{\partial x}(U - U_0) + V \frac{\partial}{\partial y}(U - U_0) + \frac{\partial}{\partial y}(\overline{uv}) = 0. \tag{4.1.22}$$

This is legitimate because U_0 is not a function of position (the imposed pressure gradient is zero). The continuity equation $\partial U/\partial x + \partial V/\partial y = 0$ may be used to rewrite the first two terms of (4.1.22) as

$$U_j \frac{\partial}{\partial x_j}(U - U_0) = \frac{\partial}{\partial x_j}[U_j(U - U_0)]. \tag{4.1.23}$$

Thus, (4.1.22) becomes

$$\frac{\partial}{\partial x}[U(U - U_0)] + \frac{\partial}{\partial y}[V(U - U_0)] + \frac{\partial}{\partial y}\overline{uv} = 0. \tag{4.1.24}$$

In jets and wakes, $U - U_0$ vanishes at sufficiently large values of y and so does $\overline{uv}$. For those flows, we may integrate (4.1.24) with respect to y over the entire flow. The result is

$$\frac{d}{dx}\int_{-\infty}^{\infty} U(U - U_0)\, dy = 0. \tag{4.1.25}$$

Consequently,

$$\rho \int_{-\infty}^{\infty} U(U - U_0)\, dy = M, \tag{4.1.26}$$

where M is a constant. This integral relation is clearly inapplicable to shear layers because their velocity defect is not integrable. For shear layers, the left-hand side of (4.1.25) is equal to $V_0 U_s$, which is unknown because V_0, the value of V at $y \to +\infty$, is unknown.

The integral (4.1.26) may be identified with the mean momentum flux across planes normal to the x axis. For wakes, $\rho(U_0 - U)$ is the net *momentum defect* per unit volume, while $U\,dy$ is the volume flux per unit depth. The integral (4.1.26) then is the net flux of momentum defect per unit depth. When we use the term *momentum defect,* we mean it in the following sense: if the wake were not present, the momentum per unit volume would be ρU_0. The difference $\rho(U_0 - U)$ is the momentum defect (or deficit). The constant M in (4.1.26) is the total momentum removed per unit time from the flow by the obstacle that produces the wake.

For jets, $U_0 = 0$, so that (4.1.26) simplifies to

$$\rho \int_{-\infty}^{\infty} U^2 \, dy = M. \tag{4.1.27}$$

Here, ρU is the mean momentum per unit volume and $U\,dy$ is the volume flux per unit depth (depth is the distance normal to the plane of the flow). Therefore, M is the total amount of momentum put into the jet at the origin per unit time.

Momentum thickness The momentum integral (4.1.26) can be used to define a length scale for turbulent wakes. Imagine that the flow past an obstacle produces a completely separated, stagnant region of width θ. The net momentum defect per unit volume is then ρU_0, because the wake contains no momentum. The total volume per unit time and depth is $U_0\theta$, so that $\rho U_0^2\,\theta$ represents the net momentum defect per unit time and depth. Thus,

$$-\rho U_0^2\,\theta = M. \tag{4.1.28}$$

Equating (4.1.26) and (4.1.28), we obtain

$$\theta = \int_{-\infty}^{\infty} \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy. \tag{4.1.29}$$

The length θ defined this way is independent of x in a plane wake; it is called the *momentum thickness* of the wake.

The momentum thickness is related to the drag coefficient of the obstacle that produces the wake. The drag coefficient c_d is defined by

$$D \equiv c_d \tfrac{1}{2} \rho U_0^2 d, \tag{4.1.30}$$

where D is the drag per unit depth and d is the frontal height of the obstacle. Clearly, $D = -M$ because the drag D produces the momentum flux M. If we equate (4.1.28) and (4.1.30), we find

$$c_d = 2\theta/d. \tag{4.1.31}$$

If the obstacle is a circular cylinder, $c_d \sim 1$ for Reynolds numbers $(U_0 d/\nu)$ between 10^3 and 3×10^5, so that θ is about $\frac{1}{2}d$ in that range.

4.2
Turbulent wakes

Here we study self-preservation (invariance), the mean momentum budget, and the kinetic energy budget of turbulence in plane wakes.

Self-preservation In the preceding analysis, we assumed that the evolution of jets, wakes, and mixing layers is determined solely by the local scales of length and velocity. Let us evaluate this assumption. In general, we may expect that in wakes

$$(U_0 - U)/U_s = f(y/\ell, \ell/L, \ell U_s/\nu, U_s/U_0). \tag{4.2.1}$$

However, we have developed approximate equations that are valid for $\ell/L \to 0$, $\ell U_s/\nu \to \infty$, $U_s/U_0 \to 0$. Under these limit processes, the (presumably monotone) dependence of the function f on ℓ/L, $\ell U_s/\nu$, and U_s/U_0 is eliminated, because no monotone function can remain finite if it does not become asymptotically independent of very large or very small parameters. Therefore, we expect that only the length scale ℓ is relevant and that all properly nondimensionalized quantities are functions of y/ℓ only. In particular,

$$(U_0 - U)/U_s = f(y/\ell), \tag{4.2.2}$$

where, of course, ℓ may change downstream ($\ell = \ell(x)$). We expect that (4.2.2) is valid because it makes a statement about velocity differences, which are related to velocity gradients. Relations like (4.2.2) do not hold for the absolute velocity U, because the value of U_0 clearly could be changed without changing the form of $U_0 - U$.

In wakes, the turbulence intensity u is of order U_s, so that we expect that the Reynolds stress may be described by

$$-\overline{uv} = U_s^2 g(y/\ell). \tag{4.2.3}$$

The set (4.2.2, 4.2.3) constitutes the *self-preservation hypothesis*: the velocity defect and the Reynolds stress become invariant with respect to x if they are expressed in terms of the local length and velocity scales ℓ and U_s.

In order to test the feasibility of (4.2.2, 4.2.3), we must substitute these expressions into the equation of motion (4.1.19). Let us define $\xi = y/\ell$, so that we may write

$$\frac{\partial U}{\partial x} = -\frac{dU_s}{dx} f + \frac{U_s}{\ell}\frac{d\ell}{dx}\xi f',$$
$$\frac{\partial \overline{uv}}{\partial y} = -\frac{U_s^2}{\ell} g', \tag{4.2.4}$$

where primes denote differentation with respect to ξ. With (4.2.4), (4.1.19) becomes

$$-\frac{U_0 \ell}{U_s^2}\frac{dU_s}{dx} f + \frac{U_0}{U_s}\frac{d\ell}{dx}\xi f' = g'. \tag{4.2.5}$$

If the shapes of f and g are to be universal, so that the normalized profiles of the velocity defect and the Reynolds stress are the same at all x, we must require that the coefficients of f and $\xi f'$ in (4.2.5) be constant. Thus, taking into account that U_0 is a constant, we need

$$\frac{\ell}{U_s^2}\frac{dU_s}{dx} = \text{const}, \quad \frac{1}{U_s}\frac{d\ell}{dx} = \text{const}. \tag{4.2.6}$$

The general solution to the pair (4.2.6) is $\ell \sim x^n$, $U_s \sim x^{n-1}$, so that another relation is needed to make the result determinate. The momentum integral (4.1.26) provides the desired constraint; using (4.2.2), we may rewrite the momentum integral as

$$U_0 U_s \ell \int_{-\infty}^{\infty} f(\xi)\, d\xi - U_s^2 \ell \int_{-\infty}^{\infty} f^2(\xi)\, d\xi = -\frac{M}{\rho}. \tag{4.2.7}$$

The second term in (4.2.7) is of order U_s/U_0 compared to the first. By virtue of (4.1.16), U_s/U_0 is of order ℓ/L, so that the second term in (4.2.7) should be neglected. Substituting for M with (4.1.28), we obtain

$$U_s \ell \int_{-\infty}^{\infty} f(\xi)\, d\xi = +U_0 \theta. \tag{4.2.8}$$

We conclude that the product $U_s\ell$ must be independent of x. If $\ell \sim x^n$ and $U_s \sim x^{n-1}$, we find that $2n-1 = 0$, so that $n = \frac{1}{2}$. Thus, ℓ and U_s are given by

$$U_s = A x^{-1/2}, \quad \ell = B x^{1/2}. \tag{4.2.9}$$

The constants A and B still have to be determined.

A self-preserving solution is possible only if the velocity and length scales behave as stated in (4.2.9). Of course, the fact that such a solution is possible does not guarantee that it occurs in nature. In many problems, possible solutions are not observed because they are not stable and change to a different form when disturbed. We need experimental evidence to determine whether or not the solution (4.3.9) indeed occurs. Experiments with plane turbulent wakes of circular cylinders have shown that the development of ℓ and U_s is well described by (4.2.9) beyond about 80 cylinder diameters. Also, measured mean-velocity profiles agree with (4.2.2) beyond $x = 80d$. However, turbulence intensities and shear stresses do not exhibit self-preservation much before $x = 200d$. In most turbulent flows the mean velocity profile reaches equilibrium long before the turbulence does. Generally, the more complicated the statistical quantity, the longer it takes to reach self-preservation. For example, $\overline{v^3}$ and $\overline{v^4}$ take longer to reach self-preservation than $\overline{v^2}$. However, all measured quantities in wakes are fully self-preserving beyond $x/d = 500$.

The mean-velocity profile If we substitute (4.2.9) into (4.2.5), we obtain

$$\tfrac{1}{2}\,(U_0 B/A)(\xi f' + f) = g'. \tag{4.2.10}$$

In order to proceed, we need a relation between f and g. If we define an eddy viscosity ν_T by $-\overline{uv} \equiv \nu_T\, \partial U/\partial y$, we can state, by virtue of (4.2.2, 4.2.3),

$$\nu_T = -U_s \ell g/f'. \tag{4.2.11}$$

Thus, we expect $\nu_T/U_s\ell$ to be some function of y/ℓ. Now, g/f' is a symmetric function, so that ν_T is approximately constant near the wake center line. Also, from physical intuition, we expect the turbulence in the wake to be thoroughly mixed, so that the scales of length and velocity should not be functions of the distance from the center line. This again suggests that ν_T may be constant.

It should be noted that (4.2.11) is a consequence of the existence of the single velocity scale U_s and the two length scales y and ℓ. Therefore, (4.2.11) is a consequence of self-preservation; it should not be construed as support for a mixing-length model. The assumption that ν_T is constant is equivalent to assuming that one of the length scales (namely y) is not relevant to ν_T.

Because both g and f' have a zero at the center line, there is some question whether ν_T remains finite as $y \to 0$. This problem is resolved with l'Hôpital's rule, which states that the limit of g/f' as $y \to 0$ is equal to the limit of g'/f''. The latter is finite at $y = 0$.

With these provisions, we proceed on the assumption that ν_T is constant:

$$\nu_T/(U_s\ell) \equiv 1/R_T = -g/f'. \tag{4.2.12}$$

The parameter $R_T \equiv U_s\ell/\nu_T$ is called the *turbulent Reynolds number*; we need experimental data to determine its value. We should keep in mind that (4.2.12) is likely to be valid only near the center line of the wake (because of symmetry); we should expect errors near the edges of the wake.

If we substitute (4.2.12) into (4.2.10), we obtain

$$\alpha(\xi f' + f) + f'' = 0, \tag{4.2.13}$$

in which

$$\alpha = \tfrac{1}{2} R_T U_0 B/A. \tag{4.2.14}$$

The solution of (4.2.13) is

$$f = \exp(-\tfrac{1}{2}\alpha \xi^2). \tag{4.2.15}$$

In accordance with the definition $U_s = \max(U_0 - U)$, we have $f(0) = 1$. We still have not defined ℓ precisely; a convenient definition is to take $\alpha = 1$ so that $f = \exp(-\frac{1}{2}) \cong 0.6$ at $\xi = 1$ $(y = \ell)$. The normalized momentum integral then becomes

$$\int_{-\infty}^{+\infty} f(\xi)\, d\xi = (2\pi)^{1/2}. \tag{4.2.16}$$

The observed value of R_T, with U_s and ℓ as previously defined, is 12.5. Substitution of (4.2.16) into (4.2.8) and of (4.2.14) (with $\alpha = 1$) into (4.2.9) then gives, with some algebra,

$$U_s/U_0 = 1.58(\theta/x)^{1/2}, \tag{4.2.17}$$

$$\ell/\theta = 0.252(x/\theta)^{1/2}. \tag{4.2.18}$$

It should be noted that the Reynolds number defined by U_s and ℓ is constant:

$$U_s\ell/\nu = 0.4\, U_0\theta/\nu. \tag{4.2.19}$$

Thus, once turbulent, a plane wake remains turbulent.

The decay laws (4.2.17) and (4.2.18) are similar to those for plane laminar wakes. This is because the momentum deficit, which is proportional to $U_s\ell$, is independent of x, so that both the Reynolds number $U_s\ell/\nu$ and the turbulent Reynolds number $U_s\ell/\nu_T$ are constant.

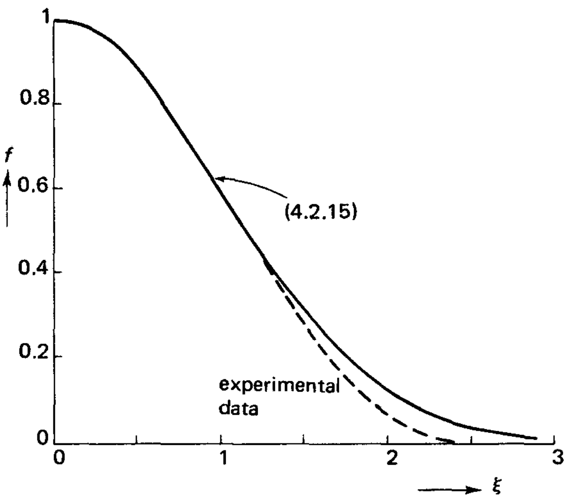

Figure 4.2. The velocity-defect profile of a plane turbulent wake (after Townsend, 1956).

The velocity profile (4.2.15) is in excellent agreement with the observed velocity profiles in wakes for all values of ξ less than 1.3. For larger values of ξ, (4.2.15) has the correct shape, but it predicts somewhat larger values of f than are observed (see Figure 4.2). The deviation is never larger than 5% of U_s. Because the predicted velocity profile (4.2.15) approaches the free-stream velocity U_0 slightly more gradually than the observations indicate, the value of ν_T appropriate for the center of the flow is evidently too large near the edges. A glance at Figure 4.3 makes the main reason for this clear. Within the turbulent part of the flow, average scales of velocity and length do not vary with cross-stream position, because there is thorough mixing from side to side. Here, a constant value of ν_T would be appropriate. Near the edges, however, a point at a fixed distance y spends only a fraction of its time in the turbulent flow. When the point is in the irrotational flow, the Reynolds stress is zero so that the net momentum transport should be multiplied by the relative fraction of time the point is in the turbulent fluid. This fraction is called the *intermittency* γ; the variation of γ is sketched in Figure 4.3. Thus, an expression like $\nu_T = \gamma \nu_{TC}$ (where ν_{TC} is the value appropriate to the center of the wake) would be a better estimate. Indeed, if a velocity profile is computed on this basis, it is found to fit the experimental data extremely well. For many purposes, however, (4.2.15) is sufficiently accurate.

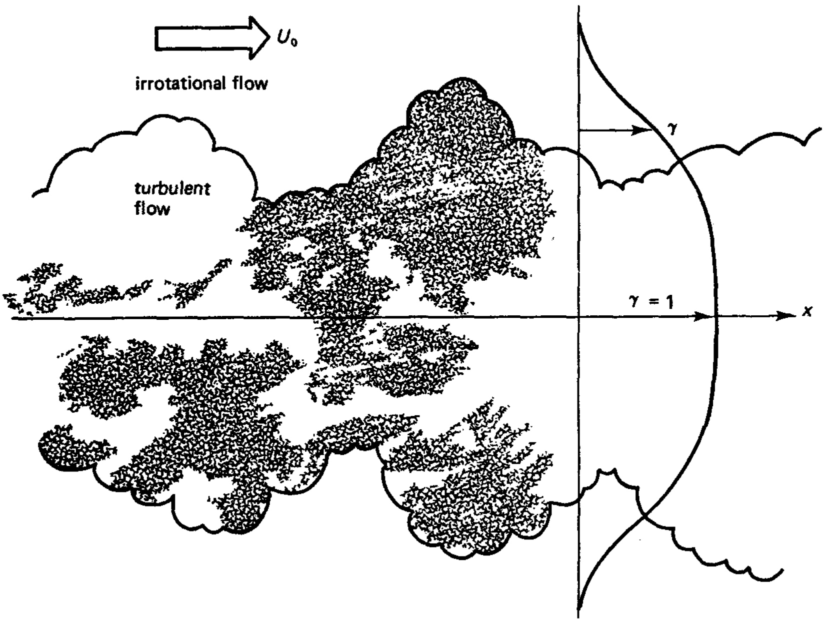

Figure 4.3. Intermittency near the edges of a wake.

Axisymmetric wakes If the foregoing analysis is applied to axisymmetric wakes, there results $U_s \sim x^{-2/3}$, $\ell \sim {}^{1/3}$, so that $R_\ell = U_s\ell/\nu \sim x^{-1/3}$. Defining U_s and ℓ in a similar way as before, we obtain $R_T = 14.1$. The structure of the axisymmetric wake is thus not likely to be markedly different from that of the plane wake, with the exception that the Reynolds number of axisymmetric wakes steadily decreases. When R_ℓ is reduced to a value of the order unity, the wake ceases to be turbulent; it develops differently as the residual velocity disturbances decay. This is not a serious practical restriction, however. Let us write

$$U_s/U_0 \sim (\theta/x)^{2/3}, \quad \ell/\theta \sim (x/\theta)^{1/3}, \tag{4.2.20}$$

and let us assume that the coefficients involved are of order unity, as they were for the plane wake. The Reynolds number R_ℓ then varies as

$$R_\ell \sim (U_0\theta/\nu)(\theta/x)^{1/3}, \tag{4.2.21}$$

so that R_ℓ reaches unity when x/θ is of order $(U_0\theta/\nu)^3$. Even for moderate Reynolds numbers this is a large distance.

Scale relations With (4.2.17, 4.2.18), we are in a position to examine quantitatively some of the scale relations in plane wakes. With the help of (4.2.3) and (4.2.12), we may write

$$-\overline{uv} = -U_s^2 f / R_T. \tag{4.2.22}$$

The Reynolds stress attains a maximum when $\xi = 1$, as differentiation of (4.2.15) (with $\alpha = 1$) shows. This yields

$$(-\overline{uv}/U_s^2)_{max} = (R_T^2 e)^{-1/2} = 0.05. \tag{4.2.23}$$

If the correlation coefficient between u and v is taken to be about 0.4, as it is in most shear flows (see Section 2.2), we obtain as an estimate for the rms velocity fluctuation $\mathscr{u}$ ($\mathscr{u}^2 = \frac{1}{3}\overline{u_i u_i} \cong \overline{u^2} \cong \overline{v^2}$):

$$\mathscr{u} \cong (0.05U_s^2/0.4)^{1/2} = 0.35U_s. \tag{4.2.24}$$

The rate at which the wake propagates into the surrounding fluid can be defined as $d\ell/dt = U_0\, d\ell/dx$, which, with (4.2.18) and (4.2.19), becomes

$$d\ell/dt = U_0\, d\ell/dx \cong 0.08\, U_s. \tag{4.2.25}$$

In a self-preserving flow we expect that all velocities are proportional to U_s, so that (4.2.24) and (4.2.25) are not surprising results. However, the values of the coefficients are interesting. The interface in Figure 4.3 propagates into the surrounding irrotational medium because it is contorted by the turbulent eddies. The contortion of the interface is caused by eddies of all scales; on the smallest scales, viscosity acts to propagate vorticity into the irrotational fluid. The net rate of propagation (or *entrainment*, as it is most often called), however, is controlled by the speed at which the contortions with the largest scales move into the surrounding fluid. Evidently, the largest eddies have a characteristic velocity roughly 0.08/0.35 ≅ 23% of that of the rms velocity fluctuation $\mathscr{u}$. This is supported by direct measurements; the large eddies contributing most to the entrainment are fairly weak, but have dimensions as large as the flow permits. They are substantially larger than the eddies that contain most of the energy.

A look at time scales is also instructive. A time scale t_p characteristic of the turbulence is given by the total energy $\frac{1}{2}\overline{u_i u_i}$ over the rate of production $-\overline{uv}\,\partial U/\partial y$ (the latter roughly equals the dissipation rate ϵ). With $\overline{u_i u_i} \cong 3\mathscr{u}^2$ and $-\overline{uv} \cong 0.4\mathscr{u}^2$, t_p becomes

$$t_\mathrm{p} \equiv \frac{\frac{1}{2}\overline{u_i u_i}}{-\overline{uv}\;\partial U/\partial y} \cong -\frac{3.75\ell}{U_\mathrm{s} f'}. \tag{4.2.26}$$

The minimum value of t_p is reached at the maximum of f', which occurs at $\xi = 1$. We obtain

$$t_\mathrm{p} \cong 6.2\,\ell/U_\mathrm{s}. \tag{4.2.27}$$

On the other hand, a time scale characteristic of the development (the downstream change) of the wake is $t_\mathrm{d} = \ell/(d\ell/dt)$, which becomes, on substitution of (4.2.25),

$$t_\mathrm{d} \equiv \ell/(d\ell/dt) \cong 12.5\,\ell/U_\mathrm{s}. \tag{4.2.28}$$

Hence, the ratio of time scales is about 2:

$$t_\mathrm{d}/t_\mathrm{p} \cong 2. \tag{4.2.29}$$

The time scale of transfer of energy to small eddies apparently is only about half the time scale of flow development. Clearly, the turbulence can never be in equilibrium because it never has time to adjust to its changing environment. The structure of turbulence in wakes can be self-preserving only because the time scale of the turbulence and that of the flow keep pace with each other as the wake moves downstream.

The turbulent energy budget The equation for the kinetic energy of the turbulence, in an approximation which is consistent with the momentum equation (4.1.19), reads

$$0 = -U_0 \frac{\partial}{\partial x}\left(\tfrac{1}{2}\overline{q^2}\right) - \overline{uv}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\overline{v\left(\tfrac{1}{2}q^2 + \frac{p}{\rho}\right)} - \epsilon. \tag{4.2.30}$$

Here, $\overline{q^2} = \overline{u_i u_i}$ is twice the kinetic energy per unit mass. The first term of (4.2.30) is convection of $\frac{1}{2}\overline{q^2}$ by the mean flow. This term is called *advection* in order to distinguish between it and thermal convection. The second term is production, the third is transport by turbulent motion, and the last is dissipation. We designate these terms by the letters *A, P, T,* and *D.*

With a few approximations, the distributions of the terms in (4.2.30) across the plane wake can be computed. We retain the approximation $-\overline{uv} = -U_\mathrm{s}^2 f'/R_\mathrm{T}$, which is known to be slightly in error toward the edges of

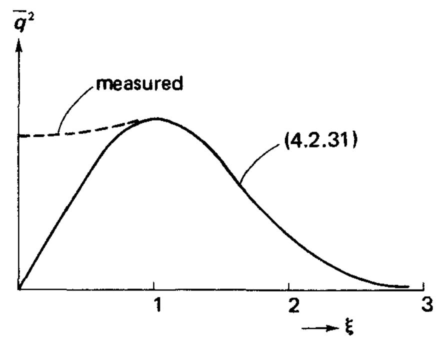

Figure 4.4. Comparison between (4.2.31) and the measured distribution of $\overline{q^2}$ in a plane wake (adapted from Townsend, 1956).

the wake. An expression for $\overline{q^2}$ is also needed. We expect that $\overline{q^2}$ and $-\overline{uv}$ are closely related; let us assume that $-\overline{uv} \cong 0.4\, \overline{q^2}/3$ outboard from the peak in f' (which occurs at $\xi = 1$). Thus for $\xi > 1$, we use

$$\overline{q^2} \cong -7.5\, U_s^2\, f'/R_T. \tag{4.2.31}$$

The region between the center line and $\xi = 1$ has to be dealt with separately, because $\overline{q^2}$ does not vanish at the center line while $-\overline{uv} = 0$ and $f' = 0$ at $\xi = 0$ for reasons of symmetry (Figure 4.4).

For the transport term we use a mixing-length assumption because it also must be self-preserving. Hence, we put

$$\overline{v\left(\frac{1}{2}q^2 + \frac{p}{\rho}\right)} = -\nu_T \frac{\partial}{\partial y}\left(\frac{1}{2}\overline{q^2}\right). \tag{4.2.32}$$

This simple form is adequate for such a crude model. We assume that ν_T is constant, realizing that this assumption is likely to be somewhat in error near the edges of the wake. Further, we take ν_T in (4.2.32) to have the same value as ν_T in (4.2.11), because the transport mechanism is probably similar. We should keep in mind that (4.2.32) cannot be applied to an off-axis peak of $\frac{1}{2}\overline{q^2}$, because we cannot use symmetry to argue for a constant (or even finite) value of ν_T.

With (4.2.31) and (4.2.32), the transport term in (4.2.30) can be expressed in terms of f. Thus we can write all terms except ϵ in terms of f. Using (4.2.15) and (4.2.17, 4.2.18), we obtain

$$\ell R_T A/U_s^3 \cong 0.3 f \xi (3 - \xi^2), \tag{4.2.33}$$

$$\ell R_T\, P/U_s^3 \cong \xi^2 f^2, \tag{4.2.34}$$

$$\ell R_T\, T/U_s^3 \cong -0.3\, f\xi(3-\xi^2). \tag{4.2.35}$$

We see that, within this approximation, the advection exactly cancels the transport, leaving the dissipation to cancel the production. The exact equality seems hardly accidental. We leave it to the reader to demonstrate that, if the exchange coefficients for momentum and energy are the same (but not necessarily constant) and if $-\overline{uv}/\overline{q^2}$ is constant, the advection and the transport always cancel, except for a term depending on the variation of R_T. The difference between advection and transport becomes smaller as the edge of the wake is approached. Also, production must be relatively small near the edge of the wake because it is quadratic in f.

The overall picture suggested by (4.2.33–4.2.35) is this: in the outer region of the wake (beyond $\xi^2 = 3$) turbulent transport brings kinetic energy from the center of the wake, where it is removed by advection. In other words, the edge of the wake is propagating into the surrounding undisturbed fluid and is blown back by the component of the mean flow normal to the wake boundary. Closer to the center, production becomes important, but it is roughly balanced by dissipation. Inboard of $\xi^2 = 3$, advection deposits kinetic energy, which is removed by transport to the outer edges of the wake. The different terms are sketched in Figure 4.5 with solid lines.

We do not expect dissipation to decrease in the center of the wake. On the contrary, we expect that the dissipation is essentially constant in the turbulent part of the flow because of the thorough mixing from one side of the wake to the other. Hence, the curve representing D should have a shape similar to that of the intermittency γ (Figure 4.3); the dissipation should decrease quite slowly from its value on the axis ($\xi = 0$) to the value $D = -P$ predicted by (4.2.34) near the production peak at $\xi = 1$. This is also sketched in Figure 4.5 with a dashed line.

The expression (4.2.34) for the production, of course, is correct near the center of the wake because $P = 0$ at $\xi = 0$. If advection, which is bringing in turbulent energy, continues to rise as the axis is approached, and if dissipation, which removes energy, does the same, while production falls off sharply, the removal of energy by turbulent transport must decrease near the axis. The decrease is somewhat delayed because the slope of A at $\xi = 1$ is larger than that of D, so that transport must increase for a while. As A and D level off, however, T must decrease. In Figure 4.5 a dashed curve represents this effect.

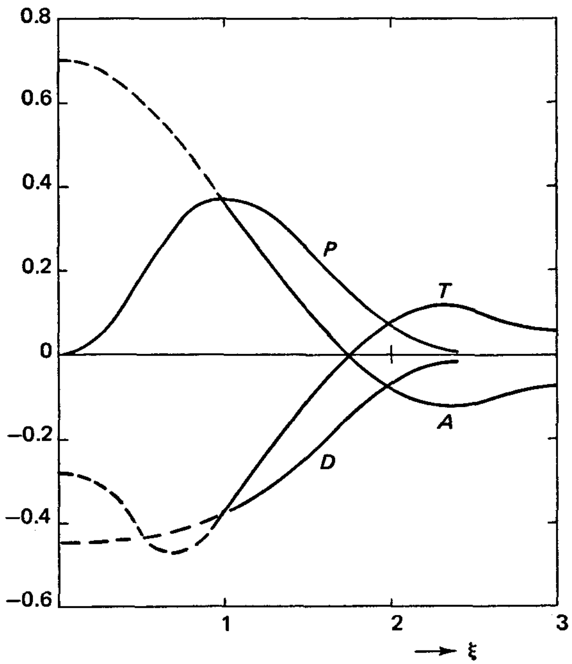

Figure 4.5. The turbulent energy budget of a wake. The solid lines are based on (4.2.33–4.2.35); the dashed lines are extrapolations described in the text.

In the central part of the wake, therefore, the mean-flow transport (advection) deposits turbulent energy, some of which is dissipated locally and some of which is transported toward the outer part of the wake. Most of the energy transported to the outer part of the wake comes from just inboard of the production peak. As an aside, we note that near the center line, gradient-transport (mixing-length) concepts are very poor: there is almost no energy gradient, $\partial(\frac{1}{2}\overline{q^2})/\partial y$, and what little there is has the wrong sign. The energy flux is locally uphill.

The predicted energy budget presented in Figure 4.5 is in good qualitative agreement with the available experimental data. However, the predicted values of advection and transport near the edge of the wake are too small by a factor of about 2. As we saw before, the measured velocity profile in wakes decreases more rapidly than the f calculated on basis of a constant eddy viscosity. Hence, the gradient of the actual f is larger than the gradient of the f that has been used in these predictions (4.2.15). If the measured velocity profile is used to calculate the advection term, it increases substantially and matches the experimental data. As we have seen, T keeps pace with A, independent of what curve is used for f, so that the use of the measured f also brings T in close agreement with the data. In effect, the predictions (4.2.33–4.2.35) should be modified for the effects of intermittency.

The fact that the dissipation decreases as fast as production near the edge of the wake is a little surprising. If $D \sim u^3/\ell$, we would expect that D would be proportional to $(f')^{3/2}/\ell$. Actually, the dissipation decreases as P, which is proportional to $(f')^2$. The explanation must be that the length scale increases as $(f')^{-1/2}$ near the outer edge. This seems realistic; as we have noted before, the eddies responsible for contorting the interface between the wake and the irrotational fluid are of larger scale.

4.3
The wake of a self-propelled body

In order to find the behavior of the length and velocity scales in self-preserving wakes, we were forced to make use of the momentum integral. In a very important practical case, that of a self-propelled body, the momentum integral vanishes. Through its propulsor (propellor, jet engine) a self-propelled body traveling at constant speed adds just enough momentum to cancel the momentum loss due to its drag, so that the wake contains no net momentum deficit. We assume that the body does not operate near an interface of two media, so that no wave drag is involved. Figure 4.6 illustrates this situation. The integral (4.2.8) vanishes identically and the value of n in $\ell \sim x^n$, $U_s \sim x^{n-1}$ remains undetermined.

It is not possible to resolve this problem without making the assumption that ν_T is constant from the beginning of the analysis. In view of the more complex structure of a self-propelled wake, with the secondary extrema of U on either side of the center line, this assumption is even more questionable than it was in the wake with nonzero momentum. For example, at the center line we have $-\overline{uv} = 0$ and $\partial U/\partial y = 0$, so that their ratio is constant because of symmetry. At the secondary extrema, however, symmetry arguments are not applicable, so that there is no reason to expect that $-\overline{uv}$ is zero where $\partial U/\partial y = 0$. All results based on a constant value of ν_T thus have a qualitative significance only. It is particularly important to recognize that the existence of similarity in wakes with finite momentum defect does not depend on the

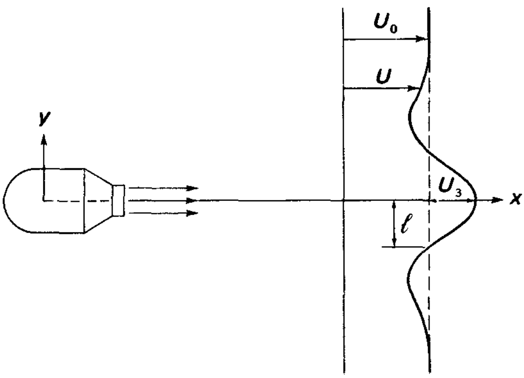

Figure 4.6. The wake of a self-propelled body. The station shown is far downstream of the body.

eddy-viscosity assumption. In the self-propelled wake, however, similarity can be obtained only by assuming that ν_T is independent of y.

Plane wakes Let us consider a plane self-propelled wake. If ν_T is independent of y, we may write the momentum equation (4.1.19) as

$$\frac{\partial}{\partial x}[U_0(U-U_0)] = \nu_T \frac{\partial^2}{\partial y^2}(U-U_0). \tag{4.3.1}$$

Here, the constant U_0 has been subtracted from U for convenience. If we multiply (4.3.1.) by y^n and integrate by parts twice, we obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty} y^n\, U_0(U-U_0)\, dy = \nu_T\, n(n-1)\int_{-\infty}^{\infty} y^{n-2}(U-U_0)\, dy. \tag{4.3.2}$$

If we put $n = 2$, the right-hand side of (4.3.2) vanishes, so that we obtain

$$\int_{-\infty}^{\infty} y^2\, U_0(U-U_0)\, dy = \text{const.} \tag{4.3.3}$$

If we further assume that the velocity-defect profile is self-preserving, there results

$$U_s \ell^3 \int_{-\infty}^{\infty} \xi^2 f(\xi)\, d\xi = \text{const.} \tag{4.3.4}$$

Because self-preservation of the equations of motion requires that $\ell \sim x^n$, $U_s \sim x^{n-1}$ (4.2.6), we obtain $3n + n - 1 = 0$ or $n = \frac{1}{4}$. Hence,

$$U_s = Ax^{-3/4}, \quad \ell = Bx^{1/4}, \tag{4.3.5}$$

where A and B are undetermined coefficients. The decay of U_s is thus substantially faster than in the wake with finite momentum.

If we substitute (4.3.5) and (4.2.12) into (4.2.5), we obtain

$$\alpha(3f + \xi f') + f'' = 0, \tag{4.3.6}$$

where $\alpha = U_0 B R_T / 4A$. The solution to (4.3.6) is

$$f = \frac{d^2}{d\xi^2}\left[\exp(-\tfrac{1}{2}\xi^2)\right]. \tag{4.3.7}$$

Here, ℓ has been defined by selecting $\alpha = 1$, as before. The velocity profile (4.3.7) is qualitatively similar to the one sketched in Figure 4.6. No information on the value of R_T in self-propelled wakes is available, although it is not likely to be much different from the value of R_T in ordinary wakes.

From an experimental point of view, it is of interest to ask what would happen if both the self-propelled and the finite-momentum wakes were simultaneously present. Imagine that a slight inaccuracy has been made in satisfying the condition of self-propulsion (zero momentum deficit). The wake then consists of

$$U_0 - U = a\exp(-\tfrac{1}{2}\xi^2) + b\frac{d^2}{d\xi^2}\left[\exp(-\tfrac{1}{2}\xi^2)\right]. \tag{4.3.8}$$

These are the first two terms of a general expansion that could be used for any wake profile (a *Gram-Charlier* expansion). Substitution of (4.3.8) and (4.2.12) in the equation of motion (4.2.5) gives, by equating like powers of ξ,

$$\ell = (2\nu_T x/U_0)^{1/2}, \quad a \propto x^{-1/2}, \quad b \propto x^{-3/2}. \tag{4.3.9}$$

This rather surprising result claims that the presence of a nonzero momentum integral dominates the growth of the length scale and forces quite rapid decay of the self-propelled component of the wake. Consider an attempt to produce a self-propelled wake in the laboratory. If we achieve self-propulsion to the extent that $b/a = 10^2$ at one body diameter (the momentum mismatch then is 1%), it takes only 10^2 body diameters downstream before the self-propelled component is overshadowed by the momentum-deficit component. This may explain why no data on self-preserving, self-propelled wakes are available. The Reynolds number of the self-propelled component of the plane, "mixed" wake varies as x^{-1}, so that this component quickly ceases to be turbulent as it progresses downstream.

Axisymmetric wakes In the case of the axisymmetric wake of a self-propelled body, an analysis similar to that just presented gives $U_s \propto x^{-4/5}$, $\ell \propto x^{1/5}$, so that $R_\ell \propto x^{-3/5}$. In the case of a "mixed" wake with self-propelled and finite-momentum components, the development of the length scale is again forced by the momentum defect, so that $\ell \propto x^{1/3}$. The momentum-defect component then decays as $x^{-2/3}$ and the self-propelled component decays as $x^{-4/3}$. Again, the Reynolds number of the self-propelled component varies as x^{-1}.

The fact that the self-propelled wake decays so much faster than the wake with finite momentum defect has some interesting implications. A maneuvering aircraft or submarine, which is accelerating or decelerating at times, leaves behind it a momentum-defect jet or wake when it is changing speed and a self-propelled wake when it is not. The latter decays much more rapidly. After some time, only the patches of wake representing changes of speed survive.

4.4
Turbulent jets and mixing layers

In jets and mixing layers there are two velocity scales, u and U_s, which are related by $u^2/U_s^2 = \mathcal{O}(\ell/L)$ as given in (4.1.20). It is clear that u/U_s needs to be constant in order to achieve self-preservation. The turbulence must retain the same relative importance as the jet develops; if the relative magnitudes of the turbulence and the mean flow are constantly changing, the flow cannot possibly be self-preserving. Because $u^2/U_s^2 = \mathcal{O}(\ell/L)$, a consequence of u/U_s being constant is that ℓ/L must be constant. Since L is a downstream length scale, we expect that in mixing layers and jets $\ell \propto x$. If ℓ/L is constant, the approximations obtained in Section 4.1 do not improve as x increases. Experiments indicate that $\ell/L \cong 6 \times 10^{-2}$, as was remarked earlier.

Because u is proportional to U_s, either one can be used as a scaling velocity. Let us use U_s, so that we can write

$$U = U_s f(\xi), \tag{4.4.1}$$

$$V = -\int_0^y \frac{\partial U}{\partial x} dy = -\ell \int_0^{\xi} \left(\frac{dU_s}{dx} f - \frac{U_s}{\ell} \frac{d\ell}{dx} \xi f' \right) d\xi, \tag{4.4.2}$$

$$-\overline{uv} = U_s^2 g(\xi), \tag{4.4.3}$$

$$\xi = y/\ell, \quad \ell = \ell(x), \quad U_s = U_s(x). \tag{4.4.4}$$

Here, as before, primes denote differentiation with respect to ξ; $\xi = 0$ at the center line. We must bear in mind that $g_{\max} \neq 1$; instead, we have $g_{\max} = u^2/U_s^2 = \mathcal{O}(\ell/L)$.

If we substitute (4.4.1–4.4.3) into (4.1.21), we obtain

$$\frac{\ell}{U_s} \frac{dU_s}{dx} f^2 - \frac{d\ell}{dx} \xi f f' - \frac{\ell}{U_s} \frac{dU_s}{dx} f' \int_0^{\xi} f\, d\xi + \frac{d\ell}{dx} f' \int_0^{\xi} \xi f'\, d\xi = g'. \tag{4.4.5}$$

Self-preservation can be obtained only if we require

$$\frac{d\ell}{dx} = A, \quad \frac{\ell}{U_s} \frac{dU_s}{dx} = B, \tag{4.4.6}$$

where A and B are constants. The first of these is not a surprise, because we already knew that $L \sim x$ and that ℓ/L must be constant. The second condition in (4.4.6) can be satisfied by any power law $U_s \propto x_n$, including $n = 0$.

Mixing layers In a mixing layer, the velocity difference U_s is imposed (Figure 4.1) by the external flow. If U_s is constant, (4.4.5) reduces to

$$-\frac{d\ell}{dx} f' \int_0^{\xi} f\, d\xi = g'. \tag{4.4.7}$$

With the eddy-viscosity assumption (4.2.12), this becomes

$$R_T \frac{d\ell}{dx} f' \int_0^{\xi} f\, d\xi = f''. \tag{4.4.8}$$

Here, of course, R_T is taken to be constant. It is not possible to obtain a solution of (4.4.8) in closed form. However, for the scale relations this is irrelevant. Let us define ℓ by taking $R_T\, d\ell/dx = 1$, so that all adjustable constants in (4.4.8) are absorbed by ℓ. This corresponds to the normalization used in wakes. The profile predicted by (4.4.8) is in fair agreement with experimental data if

$$R_T = 17.3, \quad \ell = x/17.3 = 5.7 \times 10^{-2}x. \tag{4.4.9}$$

At the edges of the mixing layer there are small discrepancies due to intermittency. The Reynolds number $U_s\ell/\nu$ of mixing layers apparently increases rapidly ($R_\ell \propto x$). Because there is no initial length (such as the jet orifice height or the momentum thickness) in the mixing layer, length scales must be compared with the viscous length ν/U_s. Experiments indicate that the mixing layer becomes self-preserving when $U_s x/\nu > 4 \times 10^5$.

Plane jets In its initial stage of development, a plane jet consists of two plane mixing layers, separated by a core of irrotational flow (Figure 4.7). Some distance after the two mixing layers have merged, the jet becomes a fully developed, self-preserving turbulent flow. The center-line velocity U_s then

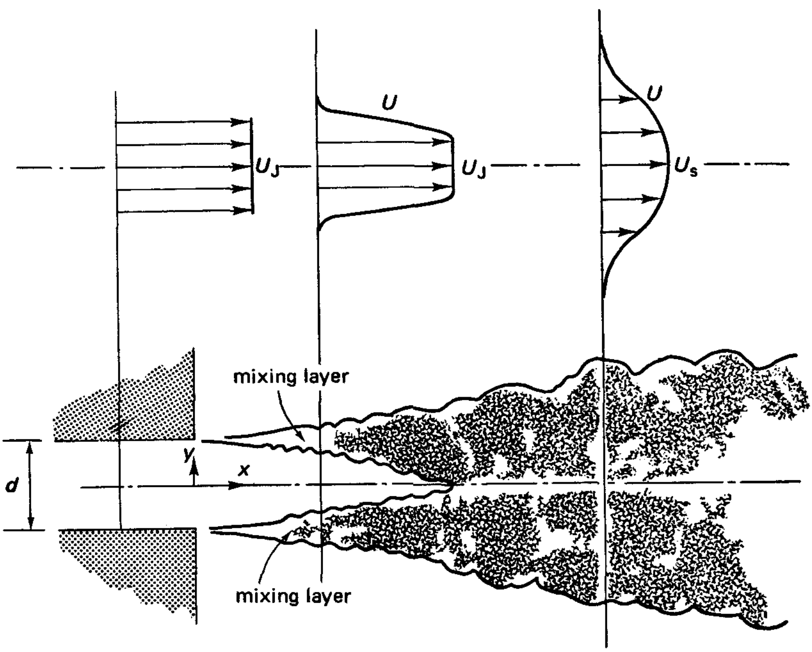

Figure 4.7. A plane turbulent jet. The jet becomes self-preserving some distance after the two mixing layers near the orifice have merged.

varies as x^n $(n \neq 0)$, and a momentum integral is needed to determine the power n. If the velocity profile is self-preserving, the momentum integral (4.1.27) becomes

$$\int_{-\infty}^{\infty} U^2\,dy = U_s^2 \ell \int_{-\infty}^{\infty} f(\xi)\,d\xi = U_J^2 d, \qquad (4.4.10)$$

where U_J is the initial jet velocity and d is the orifice height (Figure 4.7). We conclude that $2n + 1 = 0$ or $n = -\frac{1}{2}$ in order to make the momentum flux in the jet constant. Thus we obtain, for large enough values of x/d,

$$U_s/U_J = C(x/d)^{-1/2}, \qquad (4.4.11)$$

while $\ell = Ax$, as given by (4.4.6). The Reynolds number $R_\ell = U_s\ell/\nu$ increases as $x^{1/2}$, so that the viscous terms become smaller and smaller as x increases. With the use of the eddy-viscosity assumption, (4.4.5) becomes

$$-\frac{1}{2}\frac{d\ell}{dx} R_T \left(f^2 + f' \int_0^{\xi} f\,d\xi\right) = f''. \qquad (4.4.12)$$

If we define ℓ again by taking $d\ell/dx = 2/R_T$ (as in the other cases, this corresponds to $f \cong e^{-1/2}$ at $\xi = 1$), we can solve (4.4.12) to obtain

$$f = \mathrm{sech}^2 (\xi^2/2)^{1/2}. \qquad (4.4.13)$$

This fits the experimental data very well, except near the edges of the jet, if we take

$$\ell = 0.078x, \quad R_T = 25.7, \quad U_s/U_J = 2.7(d/x)^{1/2}. \qquad (4.4.14)$$

Compared with the wake, the value of R_T in jets is surprisingly large. The value of R_T in the mixing layer (4.4.9) is intermediate between those of the jet and the wake, because the mixing layer is jetlike on one side and wakelike on the other.

Not much experimental information is available on plane jets. Measured mean-velocity profiles appear to be self-preserving beyond about five orifice heights $(x/d > 5)$.

The axisymmetric jet can be approached in the same way. We obtain

$$U_s/U_J = 6.4d/x, \quad \ell = 0.067x, \quad R_T = 32. \qquad (4.4.15)$$

The Reynolds number $U_s\ell/\nu$ is constant in axisymmetric jets. No measurements have been made beyond 40 orifice diameters. The mean-velocity profile appears to be self-preserving beyond about $x/d = 8$, while the turbulence quantities are still evolving at 40 diameters.

The energy budget in a plane jet If the analysis of Section 4.1 is applied to the turbulent energy budget in a plane jet, we find that to lowest order production balances dissipation. This is too crude; if we want to take advection and transport into account, we have to include terms that are of order $(\ell/L)^{1/2}$ and ℓ/L compared to the leading terms. The full equation reads

$$0 = -U \frac{\partial}{\partial x} (\overline{\tfrac{1}{2}q^2}) - V \frac{\partial}{\partial y} (\overline{\tfrac{1}{2}q^2}) - \overline{uv} \frac{\partial U}{\partial y} - (\overline{u^2} - \overline{v^2}) \frac{\partial U}{\partial x}$$
$$- \frac{\partial}{\partial y} \overline{[(\tfrac{1}{2}q^2 + p/\rho)v]} - \epsilon. \tag{4.4.16}$$

We designate the terms by A_1, A_2, P_1, P_2, T, and D. With the same approximations as made in Section 4.2, we can obtain expressions for A_1, A_2, P_1, and T. The only term that presents a problem is P_2, which is a production term caused by normal-stress differences. On grounds of self-preservation we expect that K, defined by

$$\overline{u^2} - \overline{v^2} \equiv K(\overline{u^2} + \overline{v^2}), \tag{4.4.17}$$

is a function of $\xi = y/\ell$ only. The energy in the u component differs from that in the v component because the major production term P_1 feeds energy into $\overline{u^2}$, so that the energy must leak into $\overline{v^2}$ by inertial interaction. The value of the difference depends on the ratio of the supply rate to the leakage rate; this ratio may be expected to be constant because the two rates are determined by the same turbulence dynamics. Hence, we assume that K is not a function of position. Clearly, K is less than unity. If we use (4.2.31), (4.4.17), and the approximate relation $\overline{u^2} + \overline{v^2} \cong \frac{2}{3}q^2$, we can also express P_2 in terms of f.

Even near the edge of the jet ($\xi > 3$), we still have $y/x \ll 1$. Therefore, we do not violate the assumption of a slow evolution, and (4.1.16) remains valid. Approximate expressions for the terms in (4.4.16) near the edge of the jet ($\xi > 1$) are, if we use the mean velocity profile (4.4.13),

$$R_T \ell P_1 / U_s^3 = 2f^2, \tag{4.4.18}$$

$$R_T \ell P_2 / U_s^3 = 0.28 K f^2, \tag{4.4.19}$$

$$R_T \ell A_1 / U_s^3 = -0.58 \xi f^2, \tag{4.4.20}$$

$$R_T \ell A_2 / U_s^3 = -0.41f, \tag{4.4.21}$$

$$R_T \ell T / U_s^3 = 0.41f. \tag{4.4.22}$$

The dissipation D again can be found by difference. From (4.4.18–4.4.22) it is clear that P_1, P_2, and A_1 all are proportional to f^2, so that near the edge of the jet they become negligible long before A_2 and T, which are proportional to f. As in wakes, we find that A_2 and T have the same numerical coefficient; it can be shown that this is valid for any f if $\overline{uv}/\overline{q^2}$ is constant and if the transport term can be represented by a gradient-transport expression like (4.2.32). Thus, the energy carried by transport from the center of the jet is removed by the second advection term, A_2.

Physically, what is happening is this: near the outer edge of the jet, only one component of the mean velocity, V, is nonzero; it approaches a constant value in the plane jet, thus entraining the fluid surrounding the jet. Because the average boundary of the jet is stationary, turbulent energy must be transported into the "entrainment wind" at just that speed which keeps the average position of the interface stationary. This result is essentially independent of the assumptions embodied in (4.4.18–4.4.22). Because A_2 and T balance, dissipation plays no role. Note that A_1 plays no dominant role here, contrary to the situation in wakes.

Closer to the center line of the jet, the energy budget becomes more complicated. Calculated distributions of the terms in (4.4.16), based on the same approximations that were used for wakes, are presented in Figure 4.8. The mean velocity profile (4.4.13) was used; the second production term (P_2) has not been plotted because it is never larger than -0.003 if K in (4.4.17) is 0.4. The plot shows that A_2 and T balance in the far edge of the jet, as we discussed earlier. Somewhat closer to the center line, the sum of A_1 and A_2 approximately balances T while P and D balance each other, as in wakes. Close to the center line, A_2 becomes negligible and A_1 reverses sign. Also, P_1 must decrease to zero at $\xi = 0$ because $f' = 0$ there, and D levels out near the center line. The energy budget in the center region thus may be expected to be similar to that in the wake (Figure 4.5).

Unfortunately, there are almost no measurements with which this analysis can be compared. Near the edge of a jet, the mean velocity is small, so that the turbulence level, measured as a fraction of the mean velocity, reaches very high values, and reversal of the flow becomes a frequent occurrence. The instruments customarily used to measure turbulence (hot-wire anemometers)

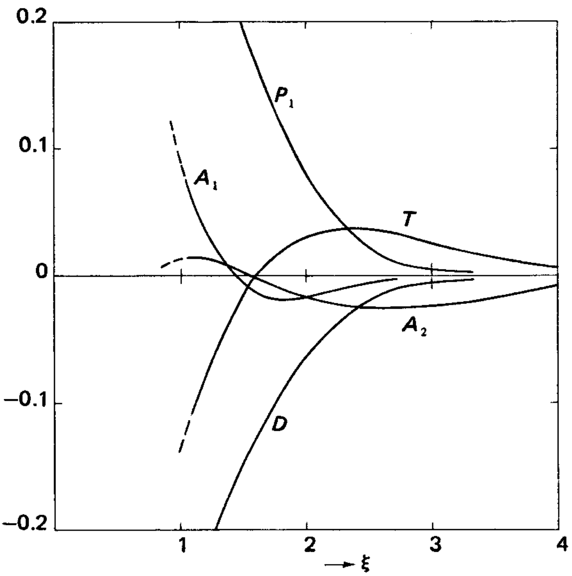

Figure 4.8. Calculated energy budget in the plane jet.

cannot tolerate this situation. However, the agreement between experimental data and predicted values is fairly good in the energy budget of a plane wake, so that we may expect that Figure 4.8, which is based on the same set of assumptions, at least presents a qualitatively correct picture.

4.5
Comparative structure of wakes, jets, and mixing layers

In Table 4.1 are collected the exponents of the power laws describing the downstream variation of U_s, ℓ, and $R_\ell = U_s\ell/\nu$ in the various flows we have examined. Also listed are the exponents, including those of the temperature scale, for buoyant plumes (Section 4.6). The values of R_T of the various flows are also listed.

The large variation in the values of R_T requires some explanation. The definition $R_T = U_s\ell/\nu_T$ uses the velocity scale U_s rather than a velocity scale characteristic of the turbulence. For jets and mixing layers, $u^2/U_s^2 \sim \ell/L$, so that the use of a suitably defined u should substantially reduce the value of R_T. Let us define a velocity scale u_* characteristic for the turbulence by

Table 4.1. Powers of x describing the downstream variation of U_s, ℓ, R_ℓ, and the temperature scale T of free shear flows. Also listed are the values of R_T and $u_*\ell_*/\nu_T$; these parameters are independent of position.

	Powers of x for					
	U_s	ℓ	R_ℓ	T	R_T	$u_*\ell_*/\nu_T$
Plane wake	−1/2	1/2	0	—	12.5	2.75
Self-propelled plane wake	−3/4	1/4	−1/2	—	?	?
Axisymmetric wake	−2/3	1/3	−1/3	—	14.1	2.92
Self-propelled axisymmetric wake	−4/5	1/5	−3/5	—	?	?
Mixing layer	0	1	1	—	17.3	4.00
Plane jet	−1/2	1	1/2	—	25.7	4.18
Axisymmetric jet	−1	1	0	—	32	4.78
Plane plume	0	1	1	−1	?	?
Axisymmetric plume	−1/3	1	2/3	−5/3	14	2.9

$$u_*^2 \equiv \max(-\overline{uv}) = \frac{U_s^2}{R_T}\max(f'). \tag{4.5.1}$$

The maximum value of f' is, of course, different in each case we have discussed. Also, the definition of ℓ varies somewhat from case to case. It would be preferable to use a length scale ℓ_* such that U_s/ℓ_* is the same fraction of the maximum of $\partial U/\partial y$; a convenient number is $e^{1/2}$, because that is the inverse of the maximum slope for plane and axisymmetric wakes. Thus,

$$\max\left(\frac{\partial U}{\partial y}\right) = \frac{U_s}{\ell}\max(f') = \frac{U_s}{\ell_*}e^{-1/2}, \tag{4.5.2}$$

which yields

$$\ell/\ell_* = e^{1/2}\max(f'). \tag{4.5.3}$$

A more meaningful turbulent Reynolds number, which allows us to compare all of the boundary-free shear flows on an equal footing, can now be defined as

$$\frac{u_*\ell_*}{\nu_T} \equiv \left\{\frac{R_T}{e\max(f')}\right\}^{1/2}. \tag{4.5.4}$$

The value of max(f') can be computed from the mean-velocity profile of each flow. If these numbers are substituted into (4.5.4), the values of $U_* \ell_* / \nu_T$ given in Table 4.1 are obtained.

The values of $u_* \ell_* / \nu_T$ clearly separate into two groups, wakes on the one hand, jets and mixing layers on the other hand. Within each group the variations in $u_* \ell_* / \nu_T$ are probably not significant, although there seems to be a consistent tendency for axisymmetric flows to have higher values than plane flows.

The difference between the two groups of flows requires explanation. The only quantity in $u_* \ell_* / \nu_T$ which is open to question is ℓ_*: it is related in a uniform way to the slope of the mean-velocity profile, but we do not know how it is related to the length scale of the turbulent eddies. Suppose that the cross-stream scales of eddies which contribute to the momentum transport in jets and mixing layers are smaller than they are in wakes. We expect that the eddy viscosity $\nu_T \sim u_* \ell_t$, where ℓ_t is a turbulence length scale. The value of $u_* \ell_* / \nu_T$ would then be effectively equal to ℓ_* / ℓ_t. In order to explain the observed difference, the value of ℓ_* / ℓ_t in jets and mixing layers needs to be about 1.5 times the value in wakes. How can we explain this?

The one important way in which jets and mixing layers differ from wakes is that the cross-stream advection term $V \, \partial U / \partial y$ is of the same order as $U \, \partial U / \partial x$ in jets and mixing layers, while the former is negligible compared to the latter in wakes. In jets and mixing layers, therefore, as much momentum is carried by the transverse flow as by the downstream flow. The transverse flow has a strain rate $(\partial V / \partial y)$ associated with it, which tends to compress eddies in the cross-stream direction. This may explain why these eddies tend to have smaller length scales than those in wakes. In fact, a crude calculation (Townsend, 1956) indicates that the expected compression factor is about 1.5. This is in agreement with observations on the intermittency γ in axisymmetric jets. The region over which γ decreases from one to zero in jets is much narrower than that in wakes, implying that the large eddies, which are responsible for contorting the interface, are indeed relatively small.

4.6
Thermal plumes

In a medium that expands on heating, a body that is hotter than its surroundings produces an upward jet of heated fluid which is driven by the density difference. The most familiar example is the plume from a cigarette in a quiet room. Atmospheric thermals rising over a surface feature of high temperature and plumes from smokestacks are other common examples. Also, if liquid of a certain density is poured into a liquid of lower density, it forms an upside-down density-driven plume. These flows can be analyzed in the same way as wakes and jets by employing the concept of self-preservation (Zel'dovitch, 1937). In the atmospheric examples we will study, we have to assume that the environment is neutrally stable. The stability plays the role of a streamwise pressure gradient; no self-preserving solutions can be expected if the stability is an arbitrary function of height.

We restrict the analysis to thermal plumes in the atmosphere, in which density differences are created by temperature differences. We use the Boussinesq approximation to the equations of motion, which was introduced in Section 3.4. We recall that in the Boussinesq approximation, the buoyancy term $-g\rho'/\bar{\rho}$ is replaced by $g\vartheta/\Theta_0$, where Θ_0 is the temperature of the adiabatic atmosphere and ϑ is the difference between the actual temperature and Θ_0. The temperature difference ϑ is decomposed into a mean value $\bar{\vartheta}$ and temperature fluctuations θ ($\bar{\theta} \equiv 0$). If $\bar{\vartheta} = 0$, the atmosphere is neutrally stable. If $\bar{\vartheta}$ increases upward, the atmosphere is stable; if it decreases upward, the atmosphere is unstable.

The Mach number of these plumes is presumed to be low, so that the continuity equation retains its customary form. If the acceleration of gravity points toward the negative x_3 direction, the equations of mean motion and mean temperature difference are

$$U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{u_i u_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} + \frac{g}{\Theta_0} \bar{\vartheta} \delta_{i3}, \tag{4.6.1}$$

$$\frac{\partial U_i}{\partial x_i} = 0, \tag{4.6.2}$$

$$U_j \frac{\partial \bar{\vartheta}}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{\theta u_j} = \gamma \frac{\partial^2 \bar{\vartheta}}{\partial x_j \partial x_j} \cdot \tag{4.6.3}$$

Two-dimensional plumes Let us consider two-dimensional plumes driven by a line source of heat (Figure 4.9). We take the z axis to be vertically upward. The line source is assumed to be parallel to the y axis, so that

$$V = 0, \quad \partial/\partial y = 0. \tag{4.6.4}$$

We assume that the flow in the plume is nearly parallel, just as in ordinary jets; we develop approximate equations of motion based on this premise. Referring to Figure 4.9, we have

$$\partial/\partial z \sim 1/L, \quad \partial/\partial x \sim 1/\ell, \quad \partial W/\partial x \sim U_s/\ell. \tag{4.6.5}$$

Substituting these estimates into (4.62), we obtain for the horizontal velocity component

$$U \sim \ell U_s/L. \tag{4.6.6}$$

We further take the turbulent velocity fluctuations to be of order u, the turbulent temperature fluctuations to be of order t, and ϑ to be of order T. The relations between these scales must be determined in the course of the analysis.

The x-momentum equation is exactly the same as (4.1.7) expressed in the proper coordinate system because the buoyancy term occurs only in the equation for the z momentum. Hence, the orders of magnitude are the same as those given in (4.1.8). Thus, with the provisions expressed in (4.1.10), (4.1.9) holds for plane plumes:

$$\rho\overline{u^2} + P = P_0. \tag{4.6.7}$$

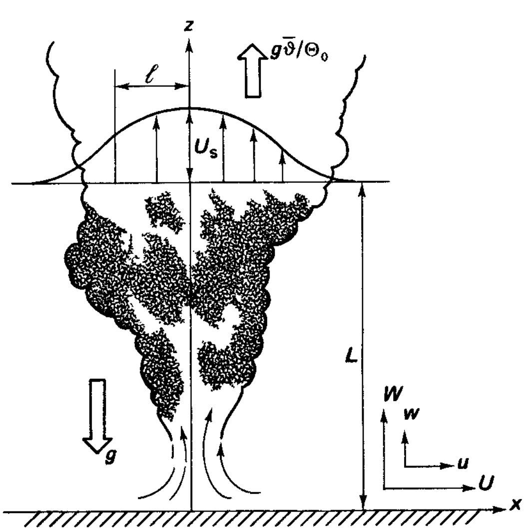

Figure 4.9. Plane thermal plume.

Substitution of (4.6.7) into the z component of (4.6.1) gives

$$U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z}+\frac{\partial}{\partial x}(\overline{uw})+\frac{\partial}{\partial z}(\overline{w^2}-\overline{u^2})$$

$$=-\frac{1}{\rho}\frac{dP_0}{dz}+\frac{g}{\Theta_0}\overline{\vartheta}+\nu\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial z^2}\right). \qquad (4.6.8)$$

Far away from the center of the plume, where there is no flow or turbulence, (4.6.8) reduces to

$$0=-\frac{1}{\rho}\frac{\partial P_0}{\partial z}+\frac{g}{\Theta_0}\overline{\vartheta}_0. \qquad (4.6.9)$$

Hence, the sum of the pressure term and the buoyancy term in (4.6.8) may be written as

$$-\frac{1}{\rho}\frac{\partial P_0}{\partial z}+\frac{g}{\Theta_0}\overline{\vartheta}=\frac{g}{\Theta_0}(\overline{\vartheta}-\overline{\vartheta}_0). \qquad (4.6.10)$$

In plane plumes, the temperature equation reads

$$U\frac{\partial\overline{\vartheta}}{\partial x}+W\frac{\partial\overline{\vartheta}}{\partial z}+\frac{\partial}{\partial x}(\overline{\theta u})+\frac{\partial}{\partial z}(\overline{\theta w})=\gamma\left(\frac{\partial^2\overline{\vartheta}}{\partial x^2}+\frac{\partial^2\overline{\vartheta}}{\partial z^2}\right). \qquad (4.6.11)$$

We have to assume that $\overline{\vartheta}_0$ is independent of z. In the momentum equation only the difference $\overline{\vartheta}-\overline{\vartheta}_0$ appears, but if the temperature equation (4.6.11) is written in terms of $\overline{\vartheta}-\overline{\vartheta}_0$, a term $W\,\partial\overline{\vartheta}_0/\partial z$ is generated, which makes self-preservation impossible. We would have additional terms on the right-hand side of (4.6.11), too, but we do not expect those to be dynamically important. If $\overline{\vartheta}_0$ is constant, the mean temperature can be written as a temperature difference everywhere in the equations, so that we lose no generality if we simply put $\overline{\vartheta}_0=0$, which means a neutral atmosphere.

The orders of magnitude of the terms in (4.6.8) are

$$U\frac{\partial W}{\partial x}:\quad \frac{U_s^2}{\ell}\frac{\ell}{L}=\left[\frac{U_s^2}{u^2}\frac{\ell}{L}\right]\frac{u^2}{\ell},$$

$$W\frac{\partial W}{\partial z}: \quad \frac{U_s^2}{L} = \left[\frac{U_s^2}{u^2}\frac{\ell}{L}\right]\frac{u^2}{\ell},$$

$$\frac{\partial \overline{uw}}{\partial x}: \quad \frac{u^2}{\ell},$$

$$\frac{\partial(\overline{w^2} - \overline{u^2})}{\partial z}: \quad \frac{u^2}{L} = \frac{\ell}{L}\cdot\frac{u^2}{\ell},$$

$$\frac{g}{\Theta_0}\overline{\vartheta}: \quad \frac{gT}{\Theta_0} = \left[\frac{gT\ell}{\Theta_0 u^2}\right]\frac{u^2}{\ell},$$

$$\nu\frac{\partial^2 W}{\partial x^2}: \quad \nu\frac{U_s}{\ell^2} = \left[R_\ell^{-1}\frac{U_s}{u}\right]\frac{u^2}{\ell},$$

$$\nu\frac{\partial^2 W}{\partial z^2}: \quad \nu\frac{U_s}{L^2} = \left[R_\ell^{-1}\frac{U_s}{u}\left(\frac{\ell}{L}\right)^2\right]\frac{u^2}{\ell}. \tag{4.6.12}$$

In order to have any mean-flow terms at all, we must again assume that

$$u/U_s = \mathcal{O}\,(\ell/L)^{1/2}. \tag{4.6.13}$$

The scaling thus is the same as in the "mechanical" jet, so that we have, to first order,

$$U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial x} + \frac{\partial}{\partial x}\overline{uw} = \frac{g}{\Theta_0}\overline{\vartheta}\,. \tag{4.6.14}$$

Note that the pressure term has been removed from (4.6.11) with (4.6.10) and $\overline{\vartheta}_0 = 0$. The temperature term has been kept in (4.6.11), although we do not know its magnitude yet. If we want thermal effects to be as important as the Reynolds stress, we need

$$gT/\Theta_0 = \mathcal{O}(u^2/\ell). \tag{4.6.15}$$

The orders of magnitude of the various terms in the temperature equation (4.6.11) become

$$U\frac{\partial\bar{\vartheta}}{\partial x}:\ \frac{T}{\ell}U_s\frac{\ell}{L}=\left[\frac{\ell}{L}\frac{T}{t}\frac{U_s}{u}\right]\frac{tu}{\ell},$$

$$W\frac{\partial\bar{\vartheta}}{\partial z}:\ \frac{T}{L}U_s=\left[\frac{\ell}{L}\frac{T}{t}\frac{U_s}{u}\right]\frac{tu}{\ell},$$

$$\frac{\partial}{\partial x}\overline{\theta u}:\frac{tu}{\ell},$$

$$\frac{\partial}{\partial z}\overline{\theta w}\ :\ \frac{tu}{L}=\frac{\ell}{L}\cdot\frac{tu}{\ell},\tag{4.6.16}$$

$$\gamma\frac{\partial^2\bar{\vartheta}}{\partial x^2}:\ \gamma\frac{T}{\ell^2}=\left[\frac{\gamma}{\nu}\frac{T}{t}\frac{1}{R_\ell}\right]\frac{tu}{\ell},$$

$$\gamma\frac{\partial^2\bar{\vartheta}}{\partial z^2}:\ \gamma\frac{T}{L^2}=\left[\frac{\gamma}{\nu}\frac{T}{t}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right)^2\right]\frac{tu}{\ell}.$$

In order to have a term which is of the same order as the third, we must require that

$$\ell T U_s=\mathcal{O}(L\,t\,u).\tag{4.6.17}$$

If the molecular diffusion terms in (4.6.12) and (4.6.16) are to be of the same order as the neglected turbulent transport terms, we need

$$\frac{U_s}{u}R_\ell^{-1}=\mathcal{O}(\ell/L),\quad \frac{\gamma}{\nu}\frac{T}{t}R_\ell^{-1}=\mathcal{O}(\ell/L).\tag{4.6.18}$$

With the aid of (4.6.13) and (4.6.17), these conditions reduce to

$$R_\ell^{-1}=\mathcal{O}(\ell/L)^{3/2},\quad (\gamma/\nu)R_\ell^{-1}=\mathcal{O}(\ell/L)^{3/2}.\tag{4.6.19}$$

In gases, $\gamma/\nu\cong 1$, so that the provisions (4.6.19) are equally stringent. Of course, if R_ℓ is larger than $(L/\ell)^{3/2}$, the molecular terms in (4.6.12, 4.6.16) are even smaller than the neglected transport terms.

With these provisions, the temperature equation reduces to

$$U\frac{\partial\bar{\vartheta}}{\partial x}+W\frac{\partial\bar{\vartheta}}{\partial z}+\frac{\partial}{\partial x}\overline{\theta u}=0.\tag{4.6.20}$$

The combination of (4.6.13) and (4.6.17) gives

$$t/T = \mathcal{O}(u/U_s) = \mathcal{O}(\ell/L)^{1/2}. \tag{4.6.21}$$

Self-preservation In order to have self-preservation, t/T and u/U_s need to be constant, because the temperature fluctuations θ should play the same relative role in the mean-temperature field $\overline{\vartheta}$ at all z, and the velocity fluctuations u,w should have the same relative importance in the mean-velocity field U,W at all z. We conclude from (4.6.21) that these requirements are consistent with the approximations developed so far if ℓ/L is a constant. Because z is the only possible choice for L, thermal plumes grow linearly ($\ell \propto z$), just like jets.

Because t/T and u/U_s are constant, we can use T and U_s as scales of temperature and velocity. The assumption of self-preservation then can be expressed as

$$W = U_s f(x/\ell) = U_s f(\xi),$$

$$U = -\ell \int_0^\xi \left(\frac{dU_s}{dz} f - \frac{U_s}{\ell} \frac{d\ell}{dx} \xi f' \right) d\xi, \tag{4.6.22}$$

$$-\overline{uw} = U_s^2 g(\xi), \quad -\overline{\theta u} = T U_s h(\xi), \quad \overline{\vartheta} = TF(\xi),$$

where $\ell = \ell(z)$, $U_s = U_s(z)$, $T = T(z)$, and $\xi = x/\ell$. If (4.6.22) is substituted into (4.6.14) and (4.6.20), there results

$$\frac{\ell}{U_s}\frac{dU_s}{dz} f' \int_0^\xi f\,d\xi + \frac{d\ell}{dz} f' \int_0^\xi \xi f'\,d\xi + \frac{\ell}{U_s}\frac{dU_s}{dz} f^2 - \frac{d\ell}{dz}\xi f f' - g'$$
$$= \frac{g}{\Theta_0}\frac{T\ell}{U_s^2} F, \tag{4.6.23}$$

$$-\frac{\ell}{U_s}\frac{dU_s}{dz} F' \int_0^\xi \xi f\,d\xi + \frac{d\ell}{dz} F' \int_0^\xi \xi f'\,d\xi + \frac{\ell}{T}\frac{dT}{dz} fF - \frac{d\ell}{dz}\xi f F' = h'. \tag{4.6.24}$$

Here the primes denote differentiation with respect to ξ. If we are to obtain self-preservation, the coefficients in (4.6.23, 4.6.24) must be constant:

$$\frac{\ell}{U_s}\frac{dU_s}{dz} = c_1, \quad \frac{d\ell}{dz} = c_2, \quad \frac{\ell}{T}\frac{dT}{dz} = c_3, \quad \frac{g}{\Theta_0}\frac{T\ell}{U_s^2} = c_4. \tag{4.6.25}$$

We clearly need linear growth of the plume; that is, $\ell = c_2 z$. The first and third relations in (4.6.25) only state that U_s and T must be powers of z. If $U_s \propto n^n$ and $T \propto z^m$, the fourth relation in (4.6.25) gives $m + 1 = 2n$, so that

$$U_s = Az^n, \quad T = Bz^{2n-1}. \tag{4.6.26}$$

We obviously need a constraint similar to a momentum integral. However, momentum is not conserved in a plume because the potential energy represented by the buoyancy is being converted into kinetic energy, so that the momentum is continually increasing. Instead, an integral related to the amount of heat added per unit time is constant.

The heat-flux integral Let us take (4.6.20) and rewrite it, with the help of the continuity equation, as

$$\frac{\partial}{\partial x}(\overline{\vartheta} U) + \frac{\partial}{\partial z}(\overline{\vartheta} W) + \frac{\partial}{\partial x}(\overline{\theta u}) = 0. \tag{4.6.27}$$

This may be integrated with respect to x, which yields

$$\int_{-\infty}^{\infty} \overline{\vartheta} W \, dx = \text{const} = \frac{H}{\rho c_p}. \tag{4.6.28}$$

The constant may be identified as $H/\rho c_p$, where H is the total heat flux in the plume, because $\rho c_p \overline{\vartheta}$ is the amount of heat per unit volume and $W\,dx$ is the volume flux per unit depth. Substituting the first and last of (4.6.22) into (4.6.28), we obtain

$$\ell T U_s \int_{-\infty}^{\infty} fF \, d\xi = \frac{H}{\rho c_p}. \tag{4.6.29}$$

Therefore, with (4.6.26) and $\ell = c_2 z$, we find

$$U_s = \text{const}, \quad T = Bz^{-1}. \tag{4.6.30}$$

If exactly the same reasoning is applied to axisymmetric plumes, we obtain

$$\ell \propto z, \quad U_s \propto z^{-1/3}, \quad T \propto z^{-5/3}. \tag{4.6.31}$$

Further results Let us return to the equations for the plane plume. Because $dU_s/dz = 0$ by virtue of (4.6.30), several terms in (4.6.23) and (4.6.24) are zero. With a little manipulation, the equation of motion reduces to

$$-\frac{d\ell}{dz} f' \int_0^\xi f \, d\xi - g' = \frac{g}{\Theta_0} \frac{T\ell}{U_s^2} F. \tag{4.6.32}$$

The simplified temperature equation can be integrated once, to yield

$$-\frac{d\ell}{dz} F \int_0^\xi f \, d\xi = h. \tag{4.6.33}$$

The presence of $d\ell/dz \sim u^2/U_s^2$ is due to the use of $-\overline{uw} = U_s^2 g$ rather than $u^2 g$: g is not of order one, but of order $d\ell/dz$.

The set (4.6.32, 4.6.33) can be solved only if the turbulent transport of momentum and heat is represented by mixing-length expressions. The eddy viscosity ν_T and the eddy thermal diffusivity γ_T may be assumed to be constant. The turbulent Prandtl number γ_T/ν_T may be taken to be equal to one, because the horizontal temperature transport depends mainly on the temperature fluctuations produced by the horizontal temperature gradient, so that temperature transport is governed by the same mechanism as momentum transport. As in "mechanical" jets, ℓ may be defined by putting $d\ell/dz \equiv 1/R_T$. No experimental data on R_T in plane plumes are available, but in axisymmetric plumes the value of R_T is about 14, with $d\ell/dz \cong 1/R_T$ if ℓ is taken as the value of x where $f \cong \exp(-\frac{1}{2})$ (Rouse, Yih, and Humphreys, 1952). This value is comparable to that in wakes, but it is substantially smaller than that in mechanical jets (Table 4.1). The entrainment wind apparently does not reduce the size of eddies in plumes. This is due to the stable temperature gradient near the plume, which compresses eddies vertically and expands them horizontally. We leave it to the reader to convince himself that this effect quantitatively tends to balance the horizontal compression caused by the entrainment wind during the life of a rising eddy.

If mixing-length expressions for $\overline{uw}$ and $\overline{\theta w}$ are substituted into (4.6.32) and (4.6.33), there results

$$-f' \int_0^\xi f \, d\xi - f'' = \frac{g}{\Theta_0} \frac{T\ell R_T}{U_s^2} F, \tag{4.6.34}$$

$$-F \int_0^\xi f \, d\xi = F'. \tag{4.6.35}$$

These equations incorporate the assumptions $d\ell/dz = 1/R_T$ and $\gamma_T = \nu_T$. At the center line of the plume, $F = 1$ *and* $f = 1$ by definition. If the shape of f may be approximated by $\exp(-\frac{1}{2}\xi^2)$, $f'' = -1$ at $\xi = 0$. At the center line, the first term of (4.6.34) vanishes, so that we obtain

$$\frac{g}{\Theta_0}\frac{T\ell R_T}{U_s^2} = 1. \tag{4.6.36}$$

The integral in (4.6.29) is about one if $F \cong f \cong \exp(-\frac{1}{2}\xi^2)$. Therefore, we obtain the approximate relation

$$\ell T U_s \cong H/\rho c_p. \tag{4.6.37}$$

From (4.6.36), (4.6.37), and $\ell = z/R_T$, we obtain

$$U_s^3 \cong \frac{gR_T H}{\Theta_0 \rho c_p}, \tag{4.6.38}$$

$$T \cong \frac{R_T H}{\rho c_p U_s z}. \tag{4.6.39}$$

With $R_T = 14$, T and U_s can be determined if the heat flux is known.

Problems

4.1 Consider an axisymmetric jet that issues from an orifice of diameter d with a velocity U_0. The ambient fluid is not at rest but moves in the same direction as the jet, with a velocity $0.1U_0$. Describe the early and late stages of development of this jet.

4.2 A very long cylinder (diameter 1 mm) is placed perpendicular to a steady airstream whose velocity is 10 m/sec. The cylinder is heated electrically; the power input is 100 watts per meter span. At what distance downstream is the rms temperature fluctuation in the wake of the cylinder reduced to 1°C? Assume that the distribution of the mean temperature difference in the wake is similar to the distribution of the mean velocity defect. For air at room temperature and pressure, $\rho = 1.25$ kg/m^3, $c_p \cong 10^3$ joule/kg°C.

4.3 A Boeing 747 taxies away from the airport gate. The pilot applies a thrust of 10,000 lb (5×10^4 newton) per engine; the engines are at a height of about 4 m above the ground. The jet exhaust is initially hot, but it rapidly cools through mixing with the ambient air. For the purposes of this problem, the initial jet velocity may be taken as the one that produces the correct amount of thrust at ambient density through the 1-m-diam engine exhaust. How far behind the engine must a 2-m-tall man stand to be reasonably sure that he will not encounter gusts (mean plus fluctuating velocities) greater than 10 m/sec? As a rule of thumb, you may assume that the probability of encountering a velocity fluctuation greater than three times the rms value is negligible.

4.4 Fresh cooling water from a nuclear power station at a river mouth is pumped out to sea in a large pipe and released at the bottom to avoid thermal pollution. Assuming that the cooling water rises as an axisymmetric density-driven plume, at what depth must the cooling water be released to avoid raising the temperature in the first 30 m below the surface by more than 1°C? The volume flow of cooling water is 10 m^3/sec; the temperature and density at the point of release are 100°C and 0.96kg/m^3, respectively. At 5°C, the density of fresh water is 1 kg/m^3, and the density of sea water is 1.03 kg/m^3.