3

THE DYNAMICS OF TURBULENCE

In Chapter 2, we studied the effects of the turbulent velocity fluctuations on the mean flow. We now turn to the other side of the issue. Two major questions arise. First, how is the kinetic energy of the turbulence maintained? Second, why are vorticity and vortex stretching so important to the study of turbulence? To help answer these questions, we shall proceed as follows. We first derive equations for the kinetic energy of the mean flow and that of the turbulence. We shall see that turbulence extracts energy from the mean flow at large scales and that this gain is approximately balanced by viscous dissipation of energy at very small scales. Realizing that dissipation of energy at small scales occurs only if there exists a dynamical mechanism that transfers [illegible]

The mean rate of strain S_{ij} is defined by

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \tag{3.1.4}$$

Since the mean momentum $\overline{u_i}$ of the turbulent velocity fluctuations is zero, we cannot discuss the effects of the mean flow on the turbulence very well in terms of mean momentum. We shall study the equations for the kinetic energy of the mean flow and of the turbulence instead. The equation governing the dynamics of the mean-flow energy $\frac{1}{2}U_iU_i$ is obtained by multiplying (3.1.1) by U_i. It is useful to split the stress term in the resulting equation into two components. The energy equation becomes

$$\rho U_j \frac{\partial}{\partial x_j}(\tfrac{1}{2} U_iU_i) = \frac{\partial}{\partial x_j}(T_{ij}U_i) - T_{ij}\frac{\partial U_i}{\partial x_j}. \tag{3.1.5}$$

Because T_{ij} is a symmetric tensor, the product $T_{ij}\, \partial U_i/\partial x_j$ is equal to the product of T_{ij} and the symmetric part S_{ij} of the deformation rate $\partial U_i/\partial x_j$; (3.1.5) thus becomes

$$\rho U_j \frac{\partial}{\partial x_j}(\tfrac{1}{2} U_iU_i) = \frac{\partial}{\partial x_j}(T_{ij}U_i) - T_{ij}S_{ij}. \tag{3.1.6}$$

The first term on the right-hand side of (3.1.6) represents transport of mean-flow energy by the stress T_{ij}. This term integrates to zero if the integration refers to a control volume on whose surface either T_{ij} or U_i vanishes. According to the divergence theorem,

$$\int_V \frac{\partial}{\partial x_j}(T_{ij}U_i)\, dV = \int_S n_j T_{ij} U_i\, ds. \tag{3.1.7}$$

The vector n_j is a unit vector normal to the surface element ds. If the work performed by the stress on the surface S of the control volume V is zero, only the volume integral of $T_{ij}S_{ij}$ can change the total amount of kinetic energy. The term $T_{ij}S_{ij}$ is called *deformation work*; by virtue of conservation of energy, it represents kinetic energy of the mean flow that is lost to or retrieved from the agency that generates the stress. The distinction between spatial energy transfer and deformation work is crucial to the understanding of the dynamics of turbulence.

Pure shear flow As an illustration, let us take a pure shear flow in which all variables depend on x_2 only and in which the only nonzero component of U_i

is U_1. For this turbulent Couette flow, which is sketched in Figure 3.1, the energy equation reads

$$0 = \frac{\partial}{\partial x_2}(T_{12}U_1) - T_{12}\frac{\partial U_1}{\partial x_2} . \qquad (3.1.8)$$

Figure 3.1 illustrates that the rate of work done by the stresses per unit volume is equal to the first term in (3.1.8). The average value of the stress is T_{12}; the work done by the average stress is equal to the second term in (3.1.8). Because the left-hand side of (3.1.8) is zero, the work $\partial(T_{12}U_1)/\partial x_2$ performed by the stresses does not result in a change of the kinetic energy of this flow; instead, it is all traded for deformation work. This is consistent with (3.1.8), because this equation implies that T_{ij} is constant. A constant stress field does not accelerate a flow; the tendency to change $\frac{1}{2}U_iU_i$ by $\partial(T_{12}U_1)/\partial x_2$ is balanced exactly by the deformation work $T_{12}\ \partial U_1/\partial x_2$. Work is performed, but $\frac{1}{2}U_iU_i$ does not change. We expect that deformation work generally will be an input term for the energy of the agency that generates the stress and that the kinetic energy $\frac{1}{2}U_iU_i$ will decrease because of the deformation work unless this loss is balanced by a net input of energy. However, no specific conclusions can be made without a study of the individual contributions of the various stresses to the deformation work.

The deformation work is caused by the stresses that contribute to T_{ij}. Substitution of (3.1.3) into $T_{ij}S_{ij}$ yields

$$T_{ij}S_{ij} = 2\mu\, S_{ij}S_{ij} - \rho\, \overline{u_iu_j}\, S_{ij}. \qquad (3.1.9)$$

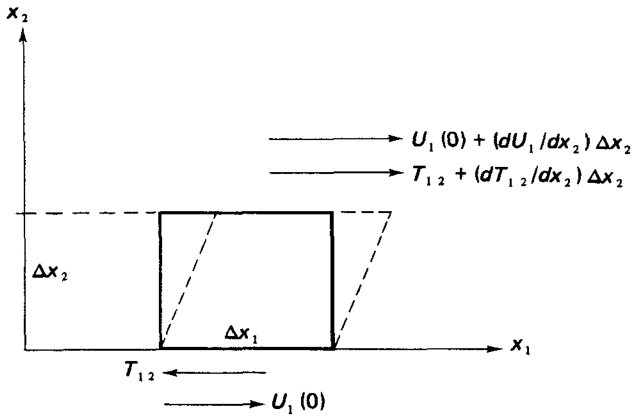


Figure 3.1. Stresses on a small volume element in a pure shear flow.

The contribution of the pressure to deformation work in an incompressible fluid is zero:

$$-P\delta_{ij}S_{ij} = -PS_{ii} = -P\frac{\partial U_i}{\partial x_i} = 0. \tag{3.1.10}$$

The contribution of viscous stresses to the deformation work is always negative; consequently, viscous deformation work always represents a loss of kinetic energy. For this reason, the term $2\mu S_{ij}S_{ij}$ is called *viscous dissipation.* Note that the dissipation is related to the strain rate, not to the vorticity (the vorticity is related to the skew-symmetric part of $\partial U_i/\partial x_j$).

The contribution of Reynolds stresses to the deformation work is also dissipative in most flows: negative values of $\overline{u_iu_j}$ tend to occur in situations with positive S_{ij}, as we have seen in Chapter 2. Positive values of $\overline{u_iu_j}\,S_{ij}$ can occur in unusual situations; even then the region in which $\overline{u_iu_j}\,S_{ij} > 0$ is a small fraction of the entire flow. Since turbulent stresses perform the deformation work, the kinetic energy of the turbulence benefits from this work. For this reason $-\rho\,\overline{u_iu_j}\,S_{ij}$ is known as *turbulent energy production.*

The effects of viscosity If (3.1.3) is substituted into (3.1.5), the energy equation for the mean flow becomes

$$U_j\frac{\partial}{\partial x_j}(\tfrac{1}{2}U_iU_i) = \frac{\partial}{\partial x_j}\left(-\frac{P}{\rho}U_j + 2\nu U_iS_{ij} - \overline{u_iu_j}\,U_i\right) + 2\nu S_{ij}S_{ij} + \overline{u_iu_j}\,S_{ij}. \tag{3.1.11}$$

The first three terms on the right-hand side of (3.1.11) are called pressure work, transport of mean-flow energy by viscous stresses, and transport of mean-flow energy by Reynolds stresses, respectively. The word "transport" refers to the integral property expressed by (3.1.7): if U_iT_{ij} is zero on the surface of a control volume, the first three terms of (3.1.11) can only redistribute energy inside the control volume.

In most flows the two viscous terms in (3.1.11) are negligible. This can be demonstrated easily by invoking the scale relation $\partial U_i/\partial x_j \sim \mathscr{u}/\ell$ (ℓ is an integral scale) and the stress estimate $-\overline{u_iu_j} \sim \mathscr{u}^2$ which were developed in Chapter 2. Of course, these relations are valid only if the turbulence is characterized by $\mathscr{u}$ and ℓ and if no other characteristic scales are present. We define the representative velocity $\mathscr{u}$ by

$$u^2 \equiv \tfrac{1}{3}\,\overline{u_i u_i}. \tag{3.1.12}$$

With $S_{ij} \sim u/\ell$ and $-\overline{u_i u_j} \sim u^2$, turbulence production is estimated as

$$\overline{u_i u_j}\,S_{ij} = C_1\, u \ell S_{ij} S_{ij}; \tag{3.1.13}$$

in the same way, energy transport by turbulent motion is estimated as

$$-\overline{u_i u_j}\,U_i = C_2\, u \ell U_i S_{ij}. \tag{3.1.14}$$

In most simple shear flows, the undetermined coefficients C_1 and C_2 are of order one. Comparing (3.1.13) and (3.1.14) with the corresponding viscous terms (3.1.11), we see that the turbulence terms are $u\ell/\nu$ times as large as the viscous terms. This Reynolds number tends to be very large (except in situations very close to smooth surfaces), so that the viscous terms in (3.1.11) can ordinarily be neglected. This conclusion again illustrates that the gross structure of turbulent flows tends to be virtually independent of viscosity. Viscosity makes itself felt only indirectly.

Although the equation for the energy of the mean flow is helpful in obtaining additional insight into the dynamics of turbulent motion, it does not contain any more information than the momentum equation for the mean flow since the former is obtained from the latter by mere manipulation.

3.2 Kinetic energy of the turbulence

The equation governing the mean kinetic energy $\frac{1}{2}\,\overline{u_i u_i}$ of the turbulent velocity fluctuations is obtained by multiplying the Navier-Stokes equations (2.1.1) by $\tilde{u}_i$, taking the time average of all terms, and subtracting (3.1.11), which governs the kinetic energy of the mean flow. This is a fairly tedious exercise, which is left to the reader. The final equation, the *turbulent energy budget*, reads

$$U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\,\overline{u_i u_i}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j p} + \tfrac{1}{2}\,\overline{u_i u_i u_j} - 2\nu\,\overline{u_i s_{ij}}\right)$$
$$-\overline{u_i u_j}\,S_{ij} - 2\nu\,\overline{s_{ij} s_{ij}}. \tag{3.2.1}$$

The quantity s_{ij} is the fluctuating rate of strain, defined by

$$s_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{3.2.2}$$

The rate of change of $\frac{1}{2}\overline{u_i u_i}$ is thus due to pressure-gradient work, transport by turbulent velocity fluctuations, transport by viscous stresses, and two kinds of deformation work. The transport terms, like those in (3.1.11), are divergences of energy flux. If the energy flux out of or into a closed control volume is zero, these terms merely redistribute energy from one point in the flow to another.

The deformation-work terms are more important. The turbulence production $-\overline{u_i u_j}\, S_{ij}$ occurs in (3.1.11) and in (3.2.1) with opposite signs. As we had anticipated, this term apparently serves to exchange kinetic energy between the mean flow and the turbulence. Normally, the energy exchange involves a loss to the mean flow and a profit to the turbulence.

The last term in (3.2.1) is the rate at which viscous stresses perform deformation work against the fluctuating strain rate. This always is a drain of energy, since the term is quadratic in s_{ij}. The term is called *viscous dissipation*; unlike the dissipation term in (3.1.11), it is essential to the dynamics of turbulence and cannot ordinarily be neglected.

Production equals dissipation In a steady, homogeneous, pure shear flow (in which all averaged quantities except U_i are independent of position and in which S_{ij} is a constant), (3.2.1) reduces to

$$-\overline{u_i u_j}\, S_{ij} = 2\nu\, \overline{s_{ij} s_{ij}}. \tag{3.2.3}$$

This equation states that in this flow the rate of production of turbulent energy by Reynolds stresses equals the rate of viscous dissipation. It should be noted that in most shear flows production and dissipation do not balance, though they are nearly always of the same order of magnitude. Keeping this in mind, we may use (3.2.3) as an aid in understanding those features of turbulence that are not directly related to spatial transport. For this reason, (3.2.3) is often written in symbolical form. If we define

$$\mathscr{P} \equiv -\overline{u_i u_j}\, S_{ij}, \tag{3.2.4}$$

$$\epsilon \equiv 2\nu\, \overline{s_{ij} s_{ij}}, \tag{3.2.5}$$

(3.2.3) reads simply

$$\mathscr{P} = \epsilon. \tag{3.2.6}$$

In order to interpret (3.2.6), we again employ the scale relation $S_{ij} \sim u/\ell$ and the stress estimate $-\overline{u_i u_j} \sim u^2$, keeping in mind that these estimates are valid only in shear-generated turbulence with one length scale and one velocity scale.

With this provision, we use (3.1.13) as an estimate for the Reynolds stress. The energy budget (3.2.3) becomes

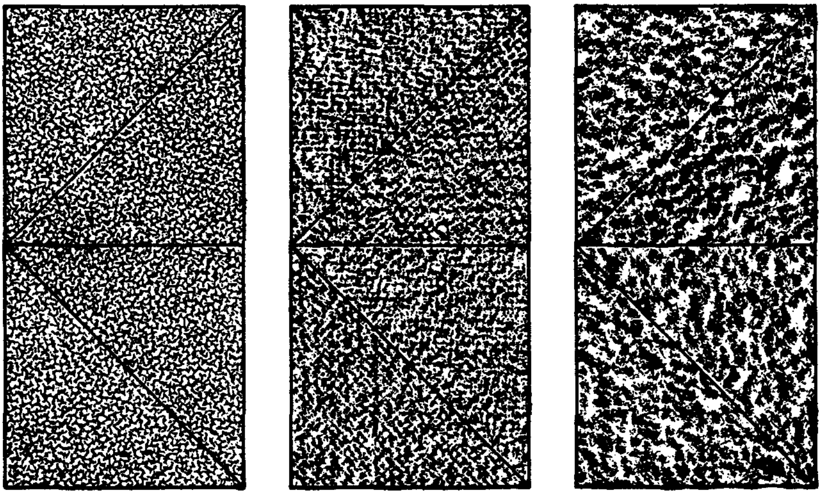
$$C_1 u \ell S_{ij} S_{ij} = 2\nu \overline{s_{ij} s_{ij}}. \tag{3.2.7}$$

Since the Reynolds number $u\ell/\nu$ is generally very large, we conclude that

$$\overline{s_{ij} s_{ij}} >> S_{ij} S_{ij}. \tag{3.2.8}$$

The fluctuating strain rate s_{ij} is thus very much larger than the mean rate of strain S_{ij} when the Reynolds number is large. Since strain rates have the dimension of $\sec^{-1}$, this implies that the eddies contributing most to the dissipation of energy have very small convective time scales compared to the time scale of the flow. This suggests that there should be very little direct interaction between the strain-rate fluctuations and the mean flow if the Reynolds number is large. In other words, S_{ij} and s_{ij} do not interact strongly, because they are not tuned to the same frequency band. Therefore, the small-scale structure of turbulence tends to be independent of any orientation effects introduced by the mean shear, so that all averages relating to the small eddies do not change under rotations or reflections of the coordinate system. If this is the case, the small-scale structure is called *isotropic* (Figure 3.2). Isotropy at small scales is called *local isotropy* (see Chapter 8).

Taylor microscale The preceding considerations suggest that any length scale involved in estimates of s_{ij} must be very much smaller than ℓ if a balance between production and dissipation is to be obtained. The situation is similar to the one in laminar boundary-layer theory (Section 1.5). In laminar boundary layers, we had to select the thickness δ in such a way that the essential viscous term in the equation of motion could be retained; this yielded $\delta/L \sim R^{-1/2}$ (1.5.3). Here, we should be able to proceed in a similar way. The dissipation of energy is proportional to $\overline{s_{ij} s_{ij}}$; this consists of several terms like $\overline{(\partial u_i/\partial x_j)^2}$, most of which cannot be measured conveniently. However, as we mentioned, the small-scale structure of turbulence tends to be



(a) (b) (c)

Figure 3.2. The shading pattern used in this book: (a) was selected because it is an isotropic random field, like the small-scale structure of turbulence. The other patterns, (b) and (c), have preferred directions; they are not isotropic.

isotropic. In isotropic turbulence, the dissipation rate is equal to

$$\epsilon = 2\nu \overline{s_{ij}s_{ij}} = 15\nu \overline{(\partial u_1/\partial x_1)^2}. \tag{3.2.9}$$

The derivation of (3.2.9) is not given here; it involves bookkeeping with terms like $\overline{(\partial u_1/\partial x_1)^2}$ that contribute to $\overline{s_{ij}s_{ij}}$ (Hinze, 1959). The coefficient 15 in (3.2.9) is considerably larger than one because so many components are involved. In many flows, $\overline{(\partial u_1/\partial x_1)^2}$ can be measured relatively easily.

Let us define a new length scale λ by

$$\overline{(\partial u_1/\partial x_1)^2} \equiv \overline{u_1^2}/\lambda^2 = u^2/\lambda^2. \tag{3.2.10}$$

The length scale λ is called the *Taylor microscale* in honor of G. I. Taylor who first defined (3.2.10). The Taylor microscale is also associated with the curvature of spatial velocity autocorrelations; this is discussed in Section 6.4. The substitution $\overline{u_1^2} = u^2$ can be made because in isotropic turbulence $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$, so that u^2, which was defined as $\frac{1}{3}\overline{u_iu_i}$, is equal to $\overline{u_1^2}$. Since the small-scale structure of turbulence at large Reynolds numbers is always approximately isotropic (see Section 8.3), we use

$$\epsilon = 15\nu\, u^2/\lambda^2, \tag{3.2.11}$$

with λ defined by (3.2.10), as a convenient estimate of ϵ.

A relation between λ and ℓ can be obtained from the simplified energy budget (3.2.3). If S_{ij} is of order u/ℓ and if $-\overline{u_i u_j}$ is of order u^2, we obtain

$$Au^3/\ell = 15\nu u^2/\lambda^2. \tag{3.2.12}$$

The ratio λ/ℓ is then given by

$$\frac{\lambda}{\ell} = \left(\frac{15}{A}\right)^{1/2} \left(\frac{u\ell}{\nu}\right)^{-1/2} = \left(\frac{15}{A}\right)^{1/2} R_\ell^{-1/2}. \tag{3.2.13}$$

In (3.2.12, 3.2.13), A is an undetermined constant, which is presumably of order one. Because in all turbulent flows $R_\ell \gg 1$, the Taylor microscale λ is always much smaller than the integral scale ℓ. Again we see that dissipation of energy is due to the small eddies of turbulence.

Scale relations The Taylor microscale λ is not the smallest length scale occurring in turbulence. The smallest scale is the Kolmogorov microscale η, which was introduced in Chapter 1:

$$\eta = (\nu^3/\epsilon)^{1/4}. \tag{3.2.14}$$

The difference between λ and η can be understood if we return to the definition (3.2.7) and the estimate (3.2.11) of the dissipation rate ϵ. The strain-rate fluctuations s_{ij} have the dimension of a frequency (sec^{-1}); the definition of ϵ thus defines a time scale associated with the dissipative structure of turbulence. Calling this time scale τ, we find that

$$\tau = (\nu/\epsilon)^{1/2}. \tag{3.2.15}$$

This time scale is identical to the one discovered by elementary considerations in Chapter 1. This is no coincidence. The dimensions of s_{ij} are such that the length scale λ was found by taking u as a velocity scale. There is no physical reason at all for this choice of characteristic velocity; the only scale that can be determined unambiguously is the time scale τ. The Taylor microscale should thus be used only in the combination (3.2.11):

$$u/\lambda = 0.26\,\tau^{-1} = 0.26\,(\epsilon/\nu)^{1/2}. \tag{3.2.16}$$

The Taylor microscale is thus not a characteristic length of the strain-rate field and does not represent any group of eddy sizes in which dissipative effects are strong. It is not a dissipation scale, because it is defined with the assistance of a velocity scale which is not relevant for the dissipative eddies. Even so, λ is used frequently because the estimate $s_{ij} \sim u/\lambda$ is often convenient. For future use, expressions relating ℓ, λ, and η are given:

$$\frac{\lambda}{\ell} = \left(\frac{15}{A}\right)^{1/2} R_\ell^{-1/2} = \frac{15}{A} R_\lambda^{-1}, \tag{3.2.17}$$

$$\frac{\lambda}{\eta} = \left(\frac{255}{A}\right)^{1/4} R_\ell^{1/4} = 15^{1/4} R_\lambda^{1/2}. \tag{3.2.18}$$

The undetermined constant A is the same as the one used in (3.2.12) and (3.2.13). The parameter R_λ is the *microscale Reynolds number*, which is defined by

$$R_\lambda \equiv u\lambda/\nu. \tag{3.2.19}$$

This Reynolds number may be interpreted as the ratio of the large-eddy time scale ℓ/u (which is proportional to λ^2/ν by virtue of (3.2.13)) and the time scale λ/u of the strain-rate fluctuations (Corrsin, 1959).

Spectral energy transfer The energy exchange between the mean flow and the turbulence is governed by the dynamics of the large eddies. This is clear from (3.2.7): large eddies contribute most to the turbulence production $\mathscr{P}$ because $\mathscr{P}$ increases with eddy size. The energy extracted by the turbulence from the mean flow thus enters the turbulence mainly at scales comparable to the integral scale ℓ.

The viscous dissipation of turbulent energy, on the other hand, occurs mainly at scales comparable to the Kolmogorov microscale η. As we found in Chapter 1, this implies that the internal dynamics of turbulence must transfer energy from large scales to small scales. All of the available experimental evidence suggests that this spectral energy transfer proceeds at a rate dictated by the energy of the large eddies (which is of order u^2) and their time scale (which is of order ℓ/u). Thus, the dissipation rate may always be estimated as

$$\epsilon = A\, u^3/\ell, \tag{3.2.20}$$

provided there exists only one characteristic length ℓ (Taylor, 1935). The estimate (3.2.20) is independent of the presence of turbulence production; (3.2.12) is thus a valid statement about the dissipation rate even if production and dissipation do not balance.

Of course, turbulence can maintain itself only if it receives a continuous supply of energy. If $-\overline{u_i u_j} S_{ij}$ is the only production term and if ϵ is estimated with (3.2.20), the approximate balance between $\mathscr{P}$ and ϵ which occurs in many turbulent shear flows may be written as

$$-\overline{u_i u_j}\, S_{ij} \sim A\, u^3/\ell. \tag{3.2.21}$$

This equation needs careful interpretation. It states that $\mathscr{P}$ must be of order u^3/ℓ if $\mathscr{P} = \epsilon$ and if ϵ is estimated by (3.2.20). This is distinct from the original interpretation of (3.2.12), which stated that $\mathscr{P}$ must be of order u^3/ℓ because $-\overline{u_i u_j}$ is of order u^2 and S_{ij} is of order u/ℓ, so that ϵ must be of order u^3/ℓ if $\mathscr{P} = \epsilon$. This discrepancy arises because the estimate $-\overline{u_i u_j} \sim u^2$ was introduced in Chapter 2 as an empirical statement without theoretical justification. This estimate now receives support from (3.2.21). With ϵ of order u^3/ℓ because spectral energy transfer is of that order and with S_{ij} of order u/ℓ because the vorticity of the large eddies is maintained by the vorticity and the strain rate of the mean flow, we conclude from (3.2.21) that $-\overline{u_i u_j}$ has to be of order u^2 if a balance between $\mathscr{P}$ and ϵ, however approximate, is to be obtained. Conversely, (3.2.21) states that a good correlation between u_i and u_j can be obtained only if S_{ij} and u/ℓ occur in the same range of frequencies.

Further estimates The orders of magnitude of the other terms of the original energy budget (3.2.1) need to be established. We shall use $s_{ij} \sim u/\lambda$ and $\lambda/\ell \sim R_\ell^{-1/2}$ wherever needed.

The pressure-work term in (3.2.1) is estimated as

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j p}\right) \sim \frac{u^3}{\ell}, \tag{3.2.22}$$

because the pressure fluctuations p should be of order ρu^2 and because the local length scale of the flow, which determines the gradients of averaged quantities, should be of the same order as the large-eddy size ℓ.

Mean transport of turbulent energy by turbulent motion is estimated as

$$-\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\,\overline{u_i u_i u_j}\right) \sim \frac{u^3}{\ell}. \tag{3.2.23}$$

It is tempting to estimate transport by viscous stresses in the following way:

$$2\nu \frac{\partial}{\partial x_j} \overline{(u_i s_{ij})} \sim \frac{\nu u^2}{\ell \lambda} \sim \frac{u^3}{\ell} R_\ell^{-1/2}. \qquad (3.2.24)$$

This estimate, however, is too large because it assumes that u_i and s_{ij} are well correlated. This is not likely because the time scale of the eddies contributing most to s_{ij} is much smaller than the time scale of the eddies contributing most to u_i. The problem can easily be resolved by substituting the definition (3.2.2) of s_{ij} into (3.2.24) and employing $\partial u_i / \partial x_i = 0$:

$$2\nu \frac{\partial}{\partial x_j} \overline{(u_i s_{ij})} = \nu \frac{\partial^2}{\partial x_j \partial x_j} (\tfrac{1}{2} \overline{u_i u_i}) + \nu \frac{\partial^2}{\partial x_i \partial x_j} \overline{u_i u_j}. \qquad (3.2.25)$$

Both terms on the right-hand side of (3.2.25) are of order $\nu u^2/\ell^2$, so that the correct estimate for the viscous transport term is

$$2\nu \frac{\partial}{\partial x_j} \overline{(u_i s_{ij})} \sim \nu \frac{u^2}{\ell^2} \sim \frac{u^3}{\ell} R_\ell^{-1}. \qquad (3.2.26)$$

Comparing (3.2.24) and (3.2.26), we see that the correlation coefficient between u_i and s_{ij} must be of order $R_\ell^{-1/2}$. The time scale of the large eddies is of order ℓ/u and the time scale of the dissipative eddies is of order λ/u. The ratio of these time scales is λ/ℓ, which is of order $R_\ell^{-1/2}$ by virtue of (3.2.17). The correlation coefficient thus scales with the ratio of the time scales involved. One might say that u_i and s_{ij} cannot interact strongly at large Reynolds numbers because they are not tuned to the same frequency range.

The estimates (3.2.22) through (3.2.26) show that only the viscous transport of turbulent energy can be neglected if the Reynolds number is large. The other transport terms are of the same order of magnitude as the production and dissipation rates, so that they need to be retained in most flows. The pressure-work term is sometimes neglected, partly because it cannot be measured and partly because p tends to be rather poorly correlated with u_i, except near a wall (Townsend, 1956). A possible explanation is that the pressure is a weighted integral of $u_i u_j$, so that its fluctuations tend to have scales that are larger than those of the velocity fluctuations.

Wind-tunnel turbulence As an application of the equations and estimates developed here, we discuss the decay of nearly homogeneous turbulence in a low-speed wind tunnel. Wind-tunnel turbulence is commonly generated by a grid or screen in a uniform flow without shear. The flow geometry is illustrated in Figure 3.3. If S_{ij} is zero, there is no turbulence production. The turbulence should then decay through viscous dissipation. This serves as a reminder that the approximation $\mathscr{P} \sim \epsilon$ is not always relevant.

If the frame of reference is chosen such that U_1 (a constant) is the only nonzero component of the mean velocity, the energy budget (3.2.1) becomes

$$U_1 \frac{\partial}{\partial x_1}\left(\tfrac{1}{2}\overline{u_i u_i}\right) = -\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\overline{u_1 p} + \tfrac{1}{2}\overline{u_i u_i u_1}\right) - \epsilon. \tag{3.2.27}$$

It has been assumed that the Reynolds number R_ℓ is so large that the viscous transport term can be neglected. The orders of magnitude of the various terms in (3.2.27) may be estimated as follows:

$$U_1 \frac{\partial}{\partial x_1}\left(\tfrac{1}{2}\overline{u_i u_i}\right) = \mathcal{O}\left(\frac{U_1}{x_1}\,u^2\right), \tag{3.2.28}$$

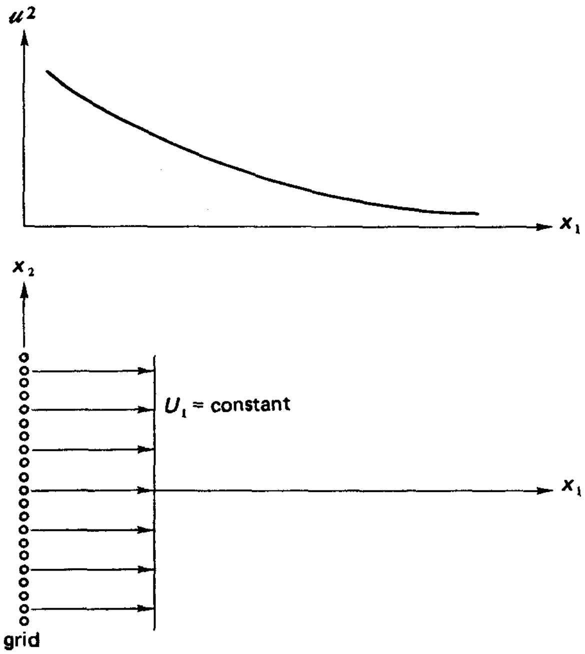


Figure 3.3. Geometry of wind-tunnel turbulence. The mean flow velocity U_1 is independent of x_1, but u^2 decreases downstream because of viscous dissipation.

$$-\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\overline{u_1 p}+\frac{1}{2}\overline{u_i u_i u_1}\right)=\mathcal{O}\left(\frac{u^3}{x_1}\right), \tag{3.2.29}$$

$$\epsilon=\left|\mathcal{O}\left(\frac{u^3}{\ell}\right)\right|. \tag{3.2.30}$$

The distance x_1 is measured from a virtual origin which is presumably in the immediate vicinity of the turbulence-producing grid. The downstream distance x_1 is the appropriate length scale in the estimate of the downstream decay of $\frac{1}{2}\overline{u_i u_i}$ and $\overline{u_1(p/\rho+\frac{1}{2}u_i u_i)}$: the integral scale ℓ is not a measure for the downstream inhomogeneity of the turbulence and no characteristic length in the downstream direction is imposed, so that $\partial/\partial x_1$ can scale only with x_1 itself. More specifically, if $u^2 \sim x_1^{\alpha}$ then $\partial u^2/dx_1 \sim \alpha u^2/x_1$, so that $\partial/\partial x_1 \sim x_1^{-1}$.

In grid turbulence, the velocity fluctuations are small: $u \ll U$. The turbulent transport terms in (3.2.27) then should be negligible compared to the transport by the mean flow, so that the energy equation reduces to

$$U_1\frac{\partial}{\partial x_1}\left(\tfrac{1}{2}\overline{u_i u_i}\right)=-\epsilon. \tag{3.2.31}$$

The dimensional estimates (3.2.28) and (3.2.30) suggest that

$$U_1/x_1=Cu/\ell, \tag{3.2.32}$$

which states that the time scale of the flow (in this case the "age" x_1/U_1 of the mean flow, which is equal to the running time on the clock of an observer moving with the mean flow) is of the same order as the time scale of the turbulence.

We would like to determine how ℓ and u change downstream. Equation (3.2.32) gives only one relation between ℓ and u in terms of x_1 and U_1, so that another relation is needed to solve this problem. Such a relation can be obtained as follows. The time scale of energy transfer from the large eddies to the small eddies is $\tau \sim \ell/u$. The time scale associated with the decay of the large eddies themselves is $T \sim \ell^2/\nu$ (based on a simple diffusion estimate like those used in Chapter 1). The ratio of these time scales is

$$T/\tau \sim \ell u/\nu, \tag{3.2.33}$$

which suggests that at large values of $R_\ell=\ell u/\nu$ the large eddies are affected very little by direct dissipation. We now assume that these time scales, together with the running time x_1/U_1, are the only independent variables of the problem. This is a fair assumption, since this flow has no time scales imposed from outside. A relation between the independent and dependent variables should exist; in nondimensional form it may be written as

$$\frac{\mathscr{u}\ell}{\nu} = f\left(\frac{x_1}{U_1\tau}, \frac{T}{\tau}\right). \tag{3.2.34}$$

Since T/τ is proportional to $\mathscr{u}\ell/\nu$, this can be rewritten

$$\frac{\mathscr{u}\ell}{\nu} = g\left(\frac{x_1}{U_1\tau}\right) = g\left(\frac{x_1\mathscr{u}}{U_1\ell}\right). \tag{3.2.35}$$

Now, the only way in which $\mathscr{u}\ell$ can be a function of $\mathscr{u}/\ell$ is by requiring that g be a constant. This is supported by the fact that the argument of g should be a constant, as predicted by (3.2.32). Hence, wind-tunnel turbulence in its initial period of decay (where $R_\ell \gg 1$) should have an approximately constant Reynolds number. Keeping in mind that R_ℓ should be independent of x_1, we find from (3.2.32)

$$\mathscr{u}^2 = \tfrac{1}{3}\overline{u_i u_i} = \mathscr{C}\frac{\nu U_1 R_\ell}{x_1}, \tag{3.2.36}$$

$$\ell = \mathscr{C}\left(\frac{x_1}{U_1}\right)^{1/2} (R_\ell \nu)^{1/2}. \tag{3.2.37}$$

The constants $\mathscr{C}$ are undetermined. Because R_ℓ is a constant, the ratios ℓ/λ and ℓ/η are constant by virtue of (3.2.17, 3.2.18). Hence λ and η also are proportional to $x_1^{1/2}$.

We conclude that the turbulent energy decays as x_1^{-1}, while all length scales grow as $x_1^{1/2}$. These results are expected to be rather crude approximations, because they are based on the assumption that only a small number of nondimensional groups is relevant. Experimental evidence indicates that the predicted exponents are within 30% of the observed values (Comte-Bellot and Corrsin, 1965). More realistic results can be obtained by spectral analysis (Problem 8.3).

At large distances from the grid, the turbulence decays much faster than indicated in the preceding analysis. The final period of decay, as this is called, cannot be understood with simple dimensional estimates, since the asymptotic behavior of the largest eddies (much larger than ℓ, but with little energy) is very complicated. The largest eddies are the ones that survive in the end; spectral analysis (Problem 8.4) is needed to resolve their decay.

Pure shear flow The energy budget of steady pure shear flow is also of interest if only because it relates to the situation discussed in Chapter 2. We adopt the notation used in that chapter: $U_1 = U_1(x_2)$, $U_2 = U_3 = 0$, $U_j\, \partial/\partial x_j = 0$, $\partial/\partial x_1 = \partial/\partial x_3 = 0$. In this flow, the only nonzero component of $\partial U_i/\partial x_j$ is $\partial U_1/\partial x_2$, so that the only nonzero components of S_{ij} are S_{12} and S_{21}, both of which are equal to $\frac{1}{2}\partial U_1/\partial x_2$. The turbulence production then is $-\overline{u_1 u_2}\, S_{12} - \overline{u_2 u_1}\, S_{21} = -\overline{u_1 u_2}\, \partial U_1/\partial x_2$. If the Reynolds number is large, the energy budget (3.2.1) reads

$$0 = -\overline{u_1 u_2}\,\frac{\partial U_1}{\partial x_2} - \frac{\partial}{\partial x_2}\left(\frac{1}{\rho}\,\overline{u_2 p} + \tfrac{1}{2}\,\overline{u_i u_i u_2}\right) - \epsilon. \tag{3.2.38}$$

All of these terms are of order u^3/ℓ; the viscous-transport term, which is much smaller (3.2.26), has been neglected.

The main features of the energy budget have already been discussed. In this simple geometry, it is worthwhile to compare (3.2.38) with the equations for the kinetic energy of the three velocity components individually. These equations are obtained in the same way as the equation for $\frac{1}{2}\overline{u_i u_i}$; if viscous transport is neglected and if the Reynolds number is so large that the dissipative structure can be assumed to be isotropic, the equations for $\frac{1}{2}\overline{u_1^2}$, $\frac{1}{2}\overline{u_2^2}$, and $\frac{1}{2}\overline{u_3^2}$ are, respectively,

$$0 = -\overline{u_1 u_2}\frac{\partial U_1}{\partial x_2} + \frac{1}{\rho}\,\overline{p\,\frac{\partial u_1}{\partial x_1}} - \frac{\partial}{\partial x_2}\,\overline{(\tfrac{1}{2}\,u_1^2 u_2)} \qquad - \tfrac{1}{3}\,\epsilon, \tag{3.2.39}$$

$$0 = \qquad 0 \qquad + \frac{1}{\rho}\,\overline{p\,\frac{\partial u_2}{\partial x_2}} - \frac{\partial}{\partial x_2}\,\overline{(p/\rho + \tfrac{1}{2}\,u_2^2)u_2} - \tfrac{1}{3}\,\epsilon, \tag{3.2.40}$$

$$0 = \qquad 0 \qquad + \frac{1}{\rho}\,\overline{p\frac{\partial u_3}{\partial x_3}} - \frac{\partial}{\partial x_2}\,\overline{(\tfrac{1}{2}\,u_3^2 u_2)} \qquad - \tfrac{1}{3}\,\epsilon. \tag{3.2.41}$$

The sum of these three equations equals (3.2.38), as it should. Note that because of incompressibility

$$\overline{p\,\frac{\partial u_1}{\partial x_1}} + \overline{p\,\frac{\partial u_2}{\partial x_2}} + \overline{p\,\frac{\partial u_3}{\partial x_3}} = \overline{p\frac{\partial u_i}{\partial x_i}} = 0. \tag{3.2.42}$$

Comparing (3.2.38) with (3.2.39–3.2.41), we see that the entire production of kinetic energy occurs in the equation for $\frac{1}{2}\overline{u_1^2}$ (3.2.39) and that the equations for $\frac{1}{2}\overline{u_2^2}$ and $\frac{1}{2}\overline{u_3^2}$ have no production terms. The u_2 and u_3 components must thus receive their energy from the pressure interaction terms listed in (3.2.42). The transport terms in (3.2.39–3.2.41) could import energy from elsewhere, but that would not explain how the u_2 and u_3 components can have energy at all: $\frac{1}{2}\overline{u_2^2}$ and $\frac{1}{2}\overline{u_3^2}$ have to be generated somehow. Because the sum of the pressure terms is equal to zero, by (3.2.42), the pressure terms exchange energy between components, without changing the total amount of energy. Also, if $\frac{1}{2}\overline{u_2^2}$ and $\frac{1}{2}\overline{u_3^2}$ are to maintain themselves, notwithstanding dissipative losses, $\overline{p\,\partial u_2/\partial x_2}$ and $\overline{p\,\partial u_3/\partial x_3}$ must be positive, so that $\overline{p\,\partial u_1/\partial x_1}$ must be negative. This, of course, can occur only if the turbulence is not isotropic. Indeed, in most shear flows $\frac{1}{2}\overline{u_1^2}$ is roughly twice as large as $\frac{1}{2}\overline{u_2^2}$ and $\frac{1}{2}\overline{u_3^2}$. In summary: the u_1 component has more energy than the other components because it receives all of the production of kinetic energy; the transfer of energy to the other components is performed by nonlinear pressure-velocity interactions.

3.3
Vorticity dynamics

All turbulent flows are characterized by high levels of fluctuating vorticity. This is the feature that distinguishes turbulence from other random fluid motions like ocean waves and atmospheric gravity waves. Therefore, we have to make a careful study of the role of vorticity fluctuations in the dynamics of turbulence.

Recalling from Section 2.3 that Reynolds stresses may be associated with eddies whose vorticity is roughly aligned with the mean strain rate, we first show that the turbulence terms in the equations for the mean flow are associated with transport and stretching of vorticity. We then turn to a study of the vorticity equation. We shall find that vorticity can indeed be amplified by line stretching due to the strain rate. The equation for the mean vorticity in a turbulent shear flow also will be explored; the interactions between velocity and vorticity fluctuations again include both transport and stretching.

Because the scale of eddies that are stretched by a strain rate decreases, the energy transfer from large eddies to small eddies may be considered in terms of vortex stretching. We shall study the mean-square vorticity fluctuations

$\overline{\omega_i\omega_i}$ in detail. The ultimate energy transfer, the dissipation of kinetic energy into heat, will turn out to be approximately equal to $\nu\,\overline{\omega_i\omega_i}$ if the Reynolds number is large. In summary, this section attempts to explain what we mean when we say that turbulence is rotational and dissipative.

Vorticity vector and rotation tensor The vorticity is the curl of the velocity vector:

$$\tilde{\omega}_i = \epsilon_{ijk}\frac{\partial \tilde{u}_k}{\partial x_j}. \tag{3.3.1}$$

This relation shows that $\tilde{\omega}_i$ is related to the deformation rate $\partial\tilde{u}_i/\partial x_j$. The deformation rate can be split up into a symmetric and a skew-symmetric part:

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{s}_{ij} + \tilde{r}_{ij}. \tag{3.3.2}$$

The strain rate $\tilde{s}_{ij}$ has been introduced before. The skew-symmetric tensor $\tilde{r}_{ij}$ is called the rotation tensor; it is defined by

$$\tilde{r}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}\right). \tag{3.3.3}$$

Since the alternating tensor ϵ_{ijk} in the definition of $\tilde{\omega}_i$ is a skew-symmetric tensor (it is +1 if i, j, k are in cyclic order, −1 if i, j, k are in anticyclic order, 0 if any two of i, j, k are equal), the vorticity vector is related only to the skew-symmetric part of $\partial\tilde{u}_i/\partial x_j$:

$$\tilde{\omega}_i = \epsilon_{ijk}\tilde{r}_{kj}. \tag{3.3.4}$$

Conversely, with some tensor algebra it is found that

$$\tilde{r}_{ij} = -\tfrac{1}{2}\,\epsilon_{ijk}\,\tilde{\omega}_k. \tag{3.3.5}$$

The one-to-one relation between the vorticity vector and the rotation tensor is due to the fact that $\tilde{r}_{ij}$ has only three independent components which, if so desired, may be represented as the components of the axial vector $\tilde{\omega}_i$.

Vortex terms in the equations of motion The vorticity equation is obtained by taking the curl of the Navier-Stokes equations. Before we perform this operation, we want to look at the way in which vorticity appears in the

Navier-Stokes equations themselves. If we treat the inertia term $\tilde{u}_j\, \partial\tilde{u}_i/\partial x_j$ as a gradient of a stress, we may write

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial}{\partial x_j}(\tilde{u}_i\tilde{u}_j) + \nu\,\frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}. \tag{3.3.6}$$

Here, the continuity equation $\partial\tilde{u}_j/\partial x_j = 0$ has been used. This particular way of writing the Navier-Stokes equations serves as a reminder that the Reynolds stress is the contribution of the velocity fluctuations to the convective terms in the equation of motion.

The convective stress term may be decomposed as follows:

$$\begin{aligned}\frac{\partial}{\partial x_j}(\tilde{u}_i\tilde{u}_j) &= \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{u}_j\left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}\right) + \tilde{u}_j\frac{\partial \tilde{u}_j}{\partial x_i}\\ &= 2\tilde{u}_j\tilde{r}_{ij} + \frac{\partial}{\partial x_i}(\tfrac{1}{2}\,\tilde{u}_j\tilde{u}_j)\\ &= -\epsilon_{ijk}\,\tilde{u}_j\,\tilde{\omega}_k + \frac{\partial}{\partial x_i}(\tfrac{1}{2}\,\tilde{u}_j\tilde{u}_j).\end{aligned} \tag{3.3.7}$$

The viscous term may be expressed in terms of vorticity by putting

$$\begin{aligned}\nu\frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} &= \nu\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}\right) + \nu\frac{\partial}{\partial x_i}\left(\frac{\partial \tilde{u}_j}{\partial x_j}\right)\\ &= 2\nu\frac{\partial}{\partial x_j}\tilde{r}_{ij} + 0\\ &= -\nu\,\epsilon_{ijk}\frac{\partial \tilde{\omega}_k}{\partial x_j}.\end{aligned} \tag{3.3.8}$$

The continuity equation $\partial u_j/\partial x_j = 0$ was used to remove the second term.

If (3.3.7) and (3.3.8) are substituted into (3.3.6), there results

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{\tilde{p}}{\rho} + \tfrac{1}{2}\,\tilde{u}_j\tilde{u}_j\right) + \epsilon_{ijk}\,\tilde{u}_j\tilde{\omega}_k - \nu\epsilon_{ijk}\frac{\partial \tilde{\omega}_k}{\partial x_j}. \tag{3.3.9}$$

In irrotational flow, $\tilde{\omega}_k = 0$ by definition, so that the viscous term and the vorticity part of the inertia term vanish. The inertia term then reduces to the gradient of the dynamic pressure $\frac{1}{2}\rho\,\tilde{u}_j\tilde{u}_j$ and (3.3.9) reduces to the Bernoulli equation. In turbulent flow, of course, neither of these conditions is satisfied.

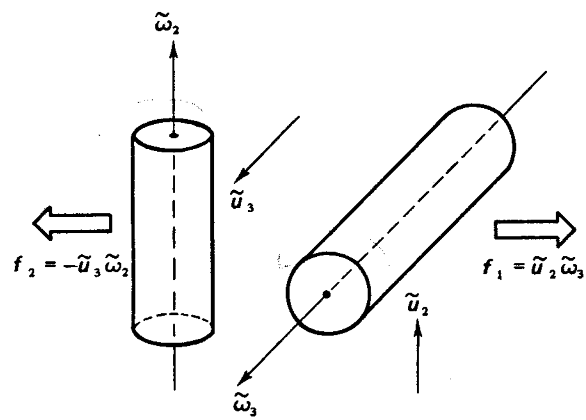


Figure 3.4. The vorticity-velocity cross product generates the body forces (per unit mass) f_1 and f_2.

The cross-product term $\epsilon_{ijk}\,\tilde{u}_j\,\tilde{\omega}_k$ is crucial to turbulence theory. It is analogous to the Coriolis force $2\,\epsilon_{ijk}\,\tilde{u}_j\,\tilde{\Omega}_k$ that would appear in the equation of motion if the coordinate system were rotating with an angular velocity $\tilde{\Omega}_k$ (the factor 2 is absent from the vorticity term because $\tilde{\omega}_k$ is twice the angular velocity of a small fluid element). The vortex term is also related to the lift force (Magnus effect) experienced by a vortex line exposed to a velocity $\tilde{u}_j$. A graphical interpretation of the "vortex force" may be helpful. In the equation for $\tilde{u}_1$, the term $\epsilon_{ijk}\,\tilde{u}_j\,\tilde{\omega}_k$ becomes $\tilde{u}_2\,\tilde{\omega}_3 - \tilde{u}_3\,\tilde{\omega}_2$. Figure 3.4 illustrates the geometry involved.

Reynolds stress and vorticity In turbulent flow, cross-product forces arise both from $U_j\,\partial U_i/\partial x_j$ and from $\partial(\overline{u_i u_j})/\partial x_j$. The instantaneous vorticity $\tilde{\omega}_i$ is decomposed into a mean vorticity Ω_i and vorticity fluctuations ω_i:

$$\tilde{\omega}_i = \Omega_i + \omega_i, \quad \overline{\omega_i} = 0. \tag{3.3.10}$$

If we assume that the flow is steady in the mean, so that we can use time averages, the equation for the mean velocity U_i may be written as

$$0 = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho} + \tfrac{1}{2}\,U_jU_j + \tfrac{1}{2}\,\overline{u_ju_j}\right) + \epsilon_{ijk}\,(U_j\Omega_k + \overline{u_j\omega_k}) + \nu\,\frac{\partial^2 U_i}{\partial x_j \partial x_j}. \tag{3.3.11}$$

Clearly, Reynolds-stress gradients contain both a dynamic-pressure gradient and an interaction term between the vorticity fluctuations and the velocity fluctuations. In many turbulent flows the contribution of the turbulence to the dynamic pressure is insignificant because $\frac{1}{2}\overline{u_j u_j} \ll \frac{1}{2}U_j U_j$. The dynamic significance of the Reynolds stress is then associated mainly with the interaction between velocity and vorticity. For a closer look at this interaction, let us consider a two-dimensional mean flow in which $U_1 \gg U_2$, $U_3 = 0$, and in which downstream derivatives of mean quantities are small compared to cross-stream derivatives ($\partial/\partial x_1 \ll \partial/\partial x_2$). This corresponds to most boundary-layer and wake flows (see Chapters 4 and 5). Under these conditions, the only nonzero component of Ω_i is $\Omega_3 = \partial U_2/\partial x_1 - \partial U_1/\partial x_2$. Because $U_2 \ll U_1$ and $\partial/\partial x_1 \ll \partial/\partial x_2$, the vorticity component Ω_3 is approximately equal to $-\partial U_1/\partial x_2$.

In the equation for U_1 the vorticity cross-product terms associated with the mean flow are $U_2\Omega_3$ and $-U_3\Omega_2$. The first of these is equal to $-U_2\,\partial U_1/\partial x_2 + U_2\,\partial U_2/\partial x_1$, the second is zero because $U_3 = 0$, $\Omega_2 = 0$. Also, $-\partial(\frac{1}{2}U_j U_j)/\partial x_1$ is equal to $-U_1\,\partial U_1/\partial x_1 - U_2\,\partial U_2/\partial x_1$ in this flow; the small term $U_2\,\partial U_2/\partial x_1$ cancels the same term generated by $U_2\Omega_3$. If we neglect the viscous term and the contribution of the turbulence to the dynamic pressure, the equation for U_1 may be written as

$$U_1\frac{\partial U_1}{\partial x_1} + U_2\frac{\partial U_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial P}{\partial x_1} + \overline{u_2\omega_3} - \overline{u_3\omega_2}. \tag{3.3.12}$$

Comparing (2.1.23) and (3.3.12) and observing that $\partial\overline{u_1^2}/\partial x_1 \ll \partial(\overline{u_1 u_2})/\partial x_2$, we find that the vortex terms represent the cross-stream derivative of the Reynolds shear stress $-\overline{u_1 u_2}$:

$$\frac{\partial}{\partial x_2}(-\overline{u_1 u_2}) = \overline{u_2\omega_3} - \overline{u_3\omega_2}. \tag{3.3.13}$$

This result can be obtained also by substituting $\omega_3 = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ and $\omega_2 = \partial u_1/\partial x_3 - \partial u_3/\partial x_1$ into $\overline{u_2\omega_3} - \overline{u_3\omega_2}$ and neglecting all terms that can be written as gradients of dynamic pressures.

Some understanding of the turbulent vorticity terms in (3.3.13) may be obtained by employing the estimate

$$-\overline{u_1 u_2} \sim \mathscr{u}\ell\,\partial U_1/\partial x_2. \tag{3.3.14}$$

If $\mathscr{u}$ is approximately independent of x_2 (this is true for many turbulent shear flows), the Reynolds-stress gradient becomes

$$\frac{\partial}{\partial x_2}(-\overline{u_1 u_2}) \sim u\ell\frac{\partial^2 U_1}{\partial x_2^2} + u\frac{\partial \ell}{\partial x_2}\frac{\partial U_1}{\partial x_2}. \tag{3.3.15}$$

Of course, (3.3.15) needs to be viewed with considerable reservation because (3.3.14) is a scaling law, not an equation. Because $\partial U_1/\partial x_2 = -\Omega_3$ approximately, (3.3.15) may be written as

$$\frac{\partial}{\partial x_2}(-\overline{u_1 u_2}) \sim -u\ell\frac{\partial \Omega_3}{\partial x_2} - u\,\Omega_3\frac{\partial \ell}{\partial x_2}. \tag{3.3.16}$$

Let us now consider $\overline{u_2\omega_3}$ and $\overline{u_3\omega_2}$. In the flow treated here, the only nonzero component of Ω_i is Ω_3. If vorticity can be transported in the x_2 direction by u_2 in the same way as momentum is transported, we should be able to write

$$\overline{u_2\omega_3} \sim -u\ell\,\partial\Omega_3/\partial x_2. \tag{3.3.17}$$

The adoption of this expression constitutes a mixing-length theory of vorticity transfer (Taylor, 1932). Of course, (3.3.17) does not need to be the same as the first term on the right-hand side of (3.3.16), because the numerical coefficients involved, which have been omitted from (3.3.16) and (3.3.17), are not necessarily equal. However, it is clear that the other term, $\overline{u_3\omega_2}$, cannot be represented by an expression like (3.3.17) because $\Omega_2 = 0$. From a comparison of (3.3.13) and (3.3.16) we conclude that the nature of $\overline{u_3\omega_2}$ is associated with a change-of-scale effect:

$$\overline{u_3\omega_2} \sim u\Omega_3\frac{\partial \ell}{\partial x_2}. \tag{3.3.18}$$

The term $\overline{u_3\omega_2}$ may be called a vortex-stretching force, since it is associated with the change of size of eddies with vorticity of order Ω_3 (see also the discussion following (3.3.35)).

The relative contributions of $\overline{u_2\omega_3}$ and $\overline{u_3\omega_2}$ to $\partial(-\overline{u_1u_2})/\partial x_2$ apparently depend on the kind of flow considered. If the length scale ℓ is approximately constant across the flow, the vortex-stretching force (3.3.18) should be negligible; the Reynolds-stress gradient may then be interpreted as vorticity transport, which should scale according to (3.3.17). This may explain why vorticity transport theory has had some success in the description of turbulent wakes and jets: in those flows, the length scale is roughly constant in the cross-stream direction.

If the length scale ℓ changes in the x_2 direction, vorticity transport theory is inadequate. A case in point is the surface layer with constant stress ($-\overline{u_1 u_2} = u_*^2$). In this flow,

$$-\overline{u_1 u_2} = u_*^2 = \kappa x_2 u_* \frac{\partial U_1}{\partial x_2}, \tag{3.3.19}$$

so that

$$\frac{\partial}{\partial x_2}(-\overline{u_1 u_2}) = 0 = \kappa x_2 u_* \frac{\partial^2 U_1}{\partial x_2^2} + \kappa u_* \frac{\partial U_1}{\partial x_2}. \tag{3.3.20}$$

According to (3.3.19), $\partial U_1/\partial x_2 = u_*/\kappa x_2$, so that $\partial^2 U_1/\partial x_2 < 0$. The vorticity-transport term $\kappa x_2 u_*\ \partial^2 U_1/\partial x_2^2 = -\kappa x_2 u_*\ \partial \Omega_3/\partial x_2$ thus is a deceleration. The deceleration of this flow is avoided because the vortex-stretching force $\kappa u_*\ \partial U_1/\partial x_2 = -\kappa u_* \Omega_3$ balances the vorticity-transport force.

One final observation needs to be made. If the local length scale of the mean-flow field is comparable to the eddy size ℓ, the order of magnitude of $\overline{u_2 \omega_3}$ and $\overline{u_3 \omega_2}$ is u^2/ℓ. Now, as we see later in this section, ω_i is of order u/λ, so that the correlation coefficient between ω_i and u_j is of order λ/ℓ. This is similar to the correlation between u_i and s_{ij} which was discussed earlier; the correlation is poor because most contributions to ω_i are made at high frequencies while most of u_j is associated with low frequencies.

The vorticity equation Let us return to the vorticity equation. This equation is obtained by applying the operator "curl" ($\epsilon_{pqi}\ \partial/\partial x_q$) to the Navier-Stokes equation (3.3.9):

$$\frac{\partial \tilde{\omega}_p}{\partial t} = -\epsilon_{pqi} \frac{\partial^2}{\partial x_i \partial x_q}\left(\frac{\tilde{p}}{\rho} + \frac{1}{2}\tilde{u}_j \tilde{u}_j\right) + (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\left(\frac{\partial}{\partial x_q}\tilde{u}_j \tilde{\omega}_k - \nu \frac{\partial^2 \tilde{\omega}_k}{\partial x_q \partial x_j}\right). \tag{3.3.21}$$

Here, the tensor identity $\epsilon_{pqi}\epsilon_{ijk} = \delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj}$ has been used. The pressure term in (3.3.21) is zero because it involves the product of the skew-symmetric tensor ϵ_{pqi} and the symmetric tensor operator $\partial^2/\partial x_i \partial x_q$. Accounting for all of the Kronecker deltas in (3.3.21), we obtain

$$\frac{\partial \tilde{\omega}_p}{\partial t} = \tilde{\omega}_k \frac{\partial \tilde{u}_p}{\partial x_k} - \tilde{u}_k \frac{\partial \tilde{\omega}_p}{\partial x_k} - \nu \frac{\partial}{\partial x_p}\left(\frac{\partial \tilde{\omega}_k}{\partial x_k}\right) + \nu \frac{\partial^2 \tilde{\omega}_p}{\partial x_k \partial x_k}. \tag{3.3.22}$$

The first of the viscous terms in (3.3.22) is zero because vorticity has zero divergence (the divergence of the curl of a vector is zero):

$$\frac{\partial \tilde{\omega}_k}{\partial x_k} = \epsilon_{ijk} \frac{\partial^2 \tilde{u}_j}{\partial x_i \partial x_k} = 0. \tag{3.3.23}$$

The final form of the vorticity equation is (changing p to i and the dummy index k to j for convenience)

$$\frac{\partial \tilde{\omega}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} = \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j \partial x_j}. \tag{3.3.24}$$

In keeping with the form of the Navier-Stokes equations introduced in Chapter 2, (3.3.24) is valid for an incompressible constant-property fluid. Before we interpret the first term on the right-hand side of (3.3.24), we want to show that the skew-symmetric part $\tilde{r}_{ij}$ of $\partial \tilde{u}_i/\partial x_j$ does not contribute to it. For this purpose, $\partial \tilde{u}_i/\partial x_j$ is split up into $\tilde{r}_{ij}$ and $\tilde{s}_{ij}$, such that

$$\tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \tilde{\omega}_j \tilde{s}_{ij} + \tilde{\omega}_j \tilde{r}_{ij}. \tag{3.3.25}$$

Because of the definition of $\tilde{r}_{ij}$, the second term in (3.3.25) becomes

$$\tilde{\omega}_j \tilde{r}_{ij} = -\tfrac{1}{2} \epsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k. \tag{3.3.26}$$

Since j and k are dummy indices they may be interchanged to yield

$$-\tfrac{1}{2} \epsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k = -\tfrac{1}{2} \epsilon_{ikj} \tilde{\omega}_j \tilde{\omega}_k. \tag{3.3.27}$$

Again interchanging the indices j and k in ϵ_{ikj}, we obtain a change in sign because ϵ_{ijk} is skew-symmetric. Hence, we find

$$-\tfrac{1}{2} \epsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k = \tfrac{1}{2} \epsilon_{ijk} \tilde{\omega}_j \tilde{\omega}_k. \tag{3.3.28}$$

This can be true only if this term is zero. Consequently, only the term in $\tilde{s}_{ij}$ survives in (3.3.25). The vorticity equation then may be written as

$$\frac{\partial \tilde{\omega}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} = \tilde{\omega}_j \tilde{s}_{ij} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j \partial x_j}. \tag{3.3.29}$$

The term $\tilde{\omega}_j \tilde{s}_{ij}$ represents amplification and rotation of the vorticity vector by the strain rate. In the context of this section, the turning of vortex axes by the strain rate is of minor importance; we shall concentrate on the components of $\tilde{\omega}_j \tilde{s}_{ij}$ that represent vortex stretching.

Vorticity apparently can be amplified by stretching of present vorticity by the strain rate $\tilde{s}_{ij}$. On the other hand, vorticity is decreased in an environment where squeezing ($\tilde{s}_{ij} < 0$) occurs.

This "source" or "sink" for vorticity is the most interesting term of the vorticity equation. It is essential to recognize that the term does not occur in two-dimensional flow. Suppose a flow is entirely in the x_1, x_2 plane. Then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are zero, so that the only nonzero vorticity component is $\tilde{\omega}_3$. The vortex-stretching term then becomes $\tilde{\omega}_3 \tilde{s}_{i3}$. However, in a two-dimensional flow only $\tilde{s}_{12}$ ($=\tilde{s}_{21}$), $\tilde{s}_{11}$, and $\tilde{s}_{22}$ can be different from zero. A two-dimensional flow cannot turn or stretch the vorticity vector.

A simple illustration of vortex stretching is the accelerated flow in a wind-tunnel contraction. Here (Figure 3.5) $\tilde{s}_{11}$ is positive, so that $\tilde{s}_{22}$ and $\tilde{s}_{33}$ must be negative to satisfy the continuity equation ($\tilde{s}_{ii} = 0$). In this kind of flow, $\tilde{\omega}_1$ is increased by vortex stretching, while $\tilde{\omega}_2$ and $\tilde{\omega}_3$ are attenuated.

The change of vorticity by vortex stretching is a consequence of the conservation of angular momentum. The angular momentum of a material

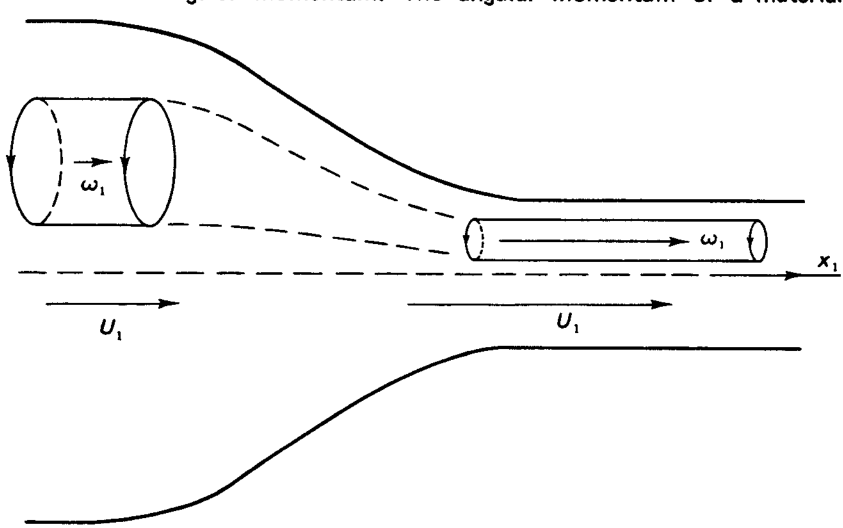


Figure 3.5. Vortex stretching in a wind-tunnel contraction. As the flow speeds up from left to right, the vorticity component ω_1 is amplified because angular momentum has to be conserved.

volume element would remain constant if viscous effects were absent; if the fluid element is stretched so that its cross-sectional area and moment of inertia become smaller, the component of the angular velocity in the direction of the stretching must increase in order to conserve angular momentum. Vortex stretching always involves a change of length scale, as Figure 3.5 illustrates. For a full account of vorticity kinematics, readers should consult general texts in fluid dynamics (for example, Batchelor, 1967).

Vorticity in turbulent flows In turbulent flow, the vorticity is decomposed into a mean vorticity Ω_i and vorticity fluctuations ω_i according to (3.3.10). After substituting (3.3.10) and the corresponding Reynolds decompositions for $\tilde{u}_i$ and $\tilde{s}_{ij}$ into (3.3.29) and taking the average of all terms in the equation, we obtain

$$U_j \frac{\partial \Omega_i}{\partial x_j} = -\overline{u_j \frac{\partial \omega_i}{\partial x_j}} + \overline{\omega_j s_{ij}} + \Omega_j S_{ij} + \nu \frac{\partial^2 \Omega_i}{\partial x_j \partial x_j} . \tag{3.3.30}$$

The mean flow has been assumed to be steady.

From (3.3.10) and (3.3.23) we conclude that both the mean vorticity and the fluctuating vorticity are solenoidal (that is, divergenceless):

$$\frac{\partial \Omega_i}{\partial x_i} = 0, \quad \frac{\partial \omega_i}{\partial x_i} = 0. \tag{3.3.31}$$

With the second equation in (3.3.31) and the continuity equation $\partial u_i/\partial x_i = 0$, the turbulence terms in (3.3.30) can be rearranged as follows:

$$\overline{u_j \frac{\partial \omega_i}{\partial x_j}} = \frac{\partial}{\partial x_j} (\overline{u_j \omega_i}), \tag{3.3.32}$$

$$\overline{\omega_j s_{ij}} = \overline{\omega_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial}{\partial x_j} (\overline{\omega_j u_i}). \tag{3.3.33}$$

The term given in (3.3.32) is clearly analogous to the Reynolds-stress term in the equation for U_i; it is due to mean transport of ω_i through its interaction with fluctuating velocities u_j in the direction of the gradients $\partial/\partial x_j$. This term, of course, changes the mean vorticity only if $\overline{u_j \omega_i}$ changes in the x_j direction. Properly speaking, (3.3.32) is a transport "divergence."

The term given in (3.3.33) is the gain (or loss) of mean vorticity caused by the stretching and rotation of fluctuating vorticity components by fluctuating strain rates.

Two-dimensional mean flow In a flow with $U_3 = 0, \Omega_1 = \Omega_2 = 0, \partial/\partial x_3 = 0$, and $\partial/\partial x_1 << \partial/\partial x_2$ (whose equation of motion was discussed earlier), the major turbulence terms in the equation for Ω_3 are

$$\overline{u_j \frac{\partial \omega_3}{\partial x_j}} \cong \frac{\partial}{\partial x_2} (\overline{u_2 \omega_3}), \tag{3.3.34}$$

$$\overline{\omega_j s_{3j}} \cong \frac{\partial}{\partial x_2} (\overline{u_3 \omega_2}). \tag{3.3.35}$$

The products $\overline{u_2 \omega_3}$ and $\overline{u_3 \omega_2}$ are related to the Reynolds-stress gradient by (3.3.13); $\overline{u_2 \omega_3}$ was interpreted as a body force arising from transport of ω_3 by u_2 in a field with a mean gradient $\partial \Omega_3/\partial x_2$, whereas $\overline{u_3 \omega_2}$ was interpreted as a body force associated with the change of size of eddies in a flow field with a varying length scale. The vortex-stretching nature of $\overline{u_3 \omega_2}$ is confirmed by (3.3.35). The cross-stream gradients of these body forces are sources or sinks for mean vorticity. In a surface layer with constant stress, the mean vorticity Ω_3 is constant along streamlines; from (3.3.17, 3.3.34) and (3.3.18, 3.3.35) we may conclude that Ω_3 is maintained because the gain of mean vorticity due to a net transport surplus is balanced by the loss of mean vorticity due to the transfer of vorticity to the turbulence by vortex stretching. A more comprehensive interpretation of (3.3.34) and (3.3.35) becomes extremely involved. Even if (3.3.17) and (3.3.18) are adopted as crude models of $\overline{u_2 \omega_3}$ and $\overline{u_3 \omega_2}$, respectively, it would be presumptuous to differentiate these equations in order to obtain models for (3.3.34, 3.3.35), because that would amount to differentiating the Reynolds-stress scaling law (3.3.14) twice. In vorticity-transfer theory, of course, the term $\overline{\omega_j s_{ij}}$ is ignored and the transport term (3.3.34) is scaled on basis of (3.3.17).

In the discussion following (3.3.20) we found that $\overline{u_2 \omega_3}$ and $\overline{u_3 \omega_2}$ both are of order u^2/ℓ. The cross-stream gradients $(\partial/\partial x_2)$ should scale with the local length scale of the mean flow, which is comparable to ℓ, in flows without multiple scales. Therefore, (3.3.34) and (3.3.35) are of order u^2/ℓ^2.

The dynamics of $\Omega_i\Omega_i$ An equation for the square of the mean vorticity is needed because the interaction between mean and fluctuating vorticities can be studied only in terms of $\Omega_i\Omega_i$ and $\overline{\omega_i\omega_i}$. Multiplying (3.3.30) by Ω_i and rearranging terms, we find

$$U_j \frac{\partial}{\partial x_j}(\tfrac{1}{2}\,\Omega_i\Omega_i) = -\frac{\partial}{\partial x_j}(\Omega_i\,\overline{\omega_i u_j}) + \overline{u_j\omega_i}\,\frac{\partial\Omega_i}{\partial x_j} + \Omega_i\Omega_j S_{ij}$$

$$+\,\Omega_i\,\overline{\omega_j s_{ij}} + \nu\frac{\partial^2}{\partial x_j\partial x_j}(\tfrac{1}{2}\,\Omega_i\Omega_i) - \nu\frac{\partial\Omega_i}{\partial x_j}\frac{\partial\Omega_i}{\partial x_j}. \qquad (3.3.36)$$

The first term on the right-hand side of (3.3.36) is the transport of $\Omega_i\Omega_i$ by turbulent vorticity-velocity interactions. This term is equivalent to the turbulent transport term of U_iU_i. The second term on the right-hand side of (3.3.36) is like the turbulence-production term in the energy equation. We may call it gradient production of $\overline{\omega_i\omega_i}$, in anticipation of the occurrence of the same term (with opposite sign) in the equation for $\overline{\omega_i\omega_i}$. The third term is stretching or shrinking of mean vorticity by the mean strain rate. The fourth term is amplification or attenuation of $\Omega_i\Omega_i$ caused by the stretching of fluctuating vorticity components by fluctuating strain rates. The fifth term is viscous transport of $\Omega_i\Omega_i$, and the sixth is viscous dissipation of $\Omega_i\Omega_i$.

The mean vorticity Ω_i is of order u/ℓ. Because $\overline{\omega_i u_j} \sim u^2/\ell$ and $\overline{\omega_j s_{ij}} \sim u^2/\ell^2$, the viscous terms in (3.3.36) are of order $(u^3/\ell^3)\,(\nu/u\ell)$, and all the other terms are of order u^3/ℓ^3. Generally speaking, therefore, only the viscous terms can be neglected. In a two-dimensional flow in the x_1, x_2 plane the only nonzero component of Ω_i is Ω_3. At large Reynolds numbers, (3.3.36) may then be approximated by

$$U_j\frac{\partial}{\partial x_j}(\tfrac{1}{2}\,\Omega_3\Omega_3) = -\frac{\partial}{\partial x_j}(\Omega_3\,\overline{\omega_3 u_j}) + \overline{u_j\omega_3}\,\frac{\partial\Omega_3}{\partial x_j} + \Omega_3\,\overline{\omega_j s_{3j}}. \qquad (3.3.37)$$

The stretching term $\Omega_i\Omega_j S_{ij}$ is zero in two-dimensional flow. If the flow involves no change of length scale, the last term of (3.3.37) may be neglected (see the discussion following (3.3.35)).

The equation for $\overline{\omega_i\omega_i}$ The equation of the mean-square vorticity fluctuations is obtained by a procedure exactly similar to the one followed for the equation of the turbulent kinetic energy. We leave the algebra as an exercise for the reader; the final result is, if the flow is steady in the mean,

$$U_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) = -\overline{u_j\omega_i}\,\frac{\partial \Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j\omega_i\omega_i}\right) + \overline{\omega_i\omega_j s_{ij}} + \overline{\omega_i\omega_j}\,S_{ij}$$

$$+\,\Omega_j\,\overline{\omega_i s_{ij}} + \nu\,\frac{\partial^2}{\partial x_j \partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) - \nu\,\overline{\frac{\partial \omega_i}{\partial x_j}\,\frac{\partial \omega_i}{\partial x_j}}\,. \qquad (3.3.38)$$

The first term on the right-hand side of (3.3.38) is the gradient production of $\overline{\omega_i\omega_i}$. This term exchanges vorticity between $\overline{\omega_i\omega_i}$ and $\Omega_i\Omega_i$ in the same way as turbulent energy production $(-\overline{u_iu_j}\,S_{ij})$ exchanges energy between U_iU_i and $\overline{u_iu_i}$.

The second term is the transport of mean-square turbulent vorticity by turbulent velocity fluctuations. This term is analogous to the transport term $\partial(\overline{u_iu_iu_j})/\partial x_j$ in the equation for $\overline{u_iu_i}$.

The third term is the production of mean-square turbulent vorticity by turbulent stretching of turbulent vorticity. We shall soon see that this is one of the dominant terms in the equation for $\overline{\omega_i\omega_i}$.

The fourth term is the production (or removal, as the case may be) of turbulent vorticity caused by the stretching (or squeezing) of vorticity fluctuations by the mean rate of strain S_{ij}.

The fifth term is a mixed production term. It occurs in the equation for $\Omega_i\Omega_i$ with the same sign. Evidently, the stretching of fluctuating vorticity by strain-rate fluctuations produces $\Omega_i\Omega_i$ and $\overline{\omega_i\omega_i}$ at the same rate.

The sixth and seventh terms on the right-hand side of (3.3.38) are viscous transport and dissipation of $\overline{\omega_i\omega_i}$, respectively.

Turbulence is rotational The equation for $\overline{\omega_i\omega_i}$ looks nearly intractable. However, if the Reynolds number is large, a very simple approximate form of (3.3.38) can be obtained, because strain-rate fluctuations are much larger than the mean strain rate and vorticity fluctuations are much larger than the mean vorticity:

$$\overline{s_{ij}s_{ij}} = \mathcal{O}\,(u/\lambda)^2, \quad S_{ij}S_{ij} = \mathcal{O}\,(u/\ell)^2, \qquad (3.3.39)$$

$$\overline{\omega_i\omega_i} = \mathcal{O}\,(u/\lambda)^2, \quad \Omega_i\Omega_i = \mathcal{O}\,(u/\ell)^2. \qquad (3.3.40)$$

As before, $\mathcal{O}$ stands for "order of magnitude." The estimates for s_{ij}, S_{ij}, and Ω_i were obtained earlier; we have to prove that the first of (3.3.40) is a valid statement before we can proceed. Some tensor algebra applied to the definitions of s_{ij}, r_{ij}, and ω_i yields

$$\overline{\omega_i\omega_i} = 2\,\overline{r_{ij}r_{ij}}, \tag{3.3.41}$$

$$\overline{s_{ij}s_{ij}} - \overline{r_{ij}r_{ij}} = \partial^2(\overline{u_iu_j})/\partial x_i\partial x_j. \tag{3.3.42}$$

Now, $\overline{s_{ij}s_{ij}}$ is of order u^2/λ^2, but the right-hand side of (3.3.42) is of order u^2/ℓ^2. Consequently, at large Reynolds numbers (3.3.42) is approximated by

$$\overline{s_{ij}s_{ij}} \cong \overline{r_{ij}r_{ij}}. \tag{3.3.43}$$

Substituting this into (3.3.41), we find

$$\overline{\omega_i\omega_i} \cong 2\,\overline{s_{ij}s_{ij}}. \tag{3.3.44}$$

From this we conclude that ω_i is of order u/λ, just like s_{ij}. This proves that the first of (3.3.40) is a valid statement if the Reynolds number is large enough. Turbulence indeed is rotational, with large vorticity fluctuations.

The strain-rate fluctuations are associated with viscous dissipation of turbulent energy. We recall that the dissipation rate ϵ is defined by

$$\epsilon \equiv 2\nu\,\overline{s_{ij}s_{ij}}. \tag{3.3.45}$$

Because of (3.3.44), this may be rewritten as

$$\epsilon \cong \nu\,\overline{\omega_i\omega_i}. \tag{3.3.46}$$

This relation shows that dissipation of energy is also associated with vorticity fluctuations. This is a useful result, but it should be kept in mind that a causal relation exists only between the strain-rate fluctuations and the dissipation rate. Indeed, (3.3.44) states merely that in flows with high Reynolds numbers the symmetric and skew-symmetric parts of the deformation-rate tensor have about the same mean-square value.

An approximate vorticity budget The estimates (3.3.39) and (3.3.40) should enable us to simplify the vorticity budget (3.3.38) appreciably. However, many of the terms in (3.3.38) contain mixed products like $\overline{\omega_iu_j}$ and $\overline{\omega_js_{ij}}$, which have to be estimated with care because they are nonzero due to the distorting effect of the mean strain rate S_{ij}. From (3.3.13) we concluded before that

$$\overline{u_i\omega_j} = \mathcal{O}(u^2/\ell); \tag{3.3.47}$$

from (3.3.13) and (3.3.33) we concluded that

$$\overline{\omega_j s_{ij}} = \mathcal{O}(u^2/\ell^2). \tag{3.3.48}$$

We also need the orders of magnitude of $\overline{\omega_i\omega_j}$ and of $\overline{u_j\omega_i\omega_i}$. The diagonal components of $\overline{\omega_i\omega_j}$ are of order u^2/λ^2, but the off-diagonal components are different from zero only in response to a mean strain rate. The mean strain rate S_{ij} is of order u/ℓ so that it can only weakly affect the vorticity structure whose characteristic frequency is u/λ. Therefore, we expect that the effect of S_{ij} should be proportional to the time-scale ratio $(\lambda/u)/(\ell/u) = \lambda/\ell$:

$$\overline{\omega_i\omega_j} = \frac{u^2}{\lambda^2}\left(a\delta_{ij} + b_{ij}\frac{\lambda}{\ell} + \ldots\right). \tag{3.3.49}$$

The coefficients a and b_{ij} should be of order one. The discount for the time-scale ratio λ/ℓ applied here is analogous to the discount needed in $\overline{u_i\omega_j}$. The term $\overline{\omega_i\omega_j}\,S_{ij}$ in (3.3.38) becomes

$$\overline{\omega_i\omega_j}\;S_{ij} = \frac{u^2}{\lambda^2}\left(aS_{ii} + b_{ij}\frac{\lambda}{\ell}S_{ij} + \ldots\right). \tag{3.3.50}$$

Because $S_{ii} = 0$ as a result of incompressibility, and $b_{ij}\,S_{ij} \sim u/\ell$, we find that

$$\overline{\omega_i\omega_j}\;S_{ij} = \mathcal{O}(u^3/\lambda\ell^2). \tag{3.3.51}$$

The transport term $\partial(\overline{u_j\omega_i\omega_i})/\partial x_j$ may be written as

$$\frac{\partial}{\partial x_j}\overline{u_j\omega_i\omega_i} = \overline{u_j\frac{\partial}{\partial x_j}(\omega_i\omega_i)}. \tag{3.3.52}$$

This term does not depend on the mean strain rate but on inhomogeneity in the distribution of mean square vorticity. If we assume that turbulent motion is an effective "mixer" of vorticity, u_j should be well correlated with the gradients of $\overline{\omega_i\omega_i}$, so that

$$\frac{\partial}{\partial x_j}(\overline{u_j\omega_i\omega_i}) = \mathcal{O}\left(\frac{u}{\ell}\cdot\frac{u^2}{\lambda^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^2\ell}\right). \tag{3.3.53}$$

With the results obtained above, most of the terms of (3.3.38) can be estimated. We obtain

$$-\overline{u_j\omega_i}\frac{\partial\Omega_i}{\partial x_j} = \mathcal{O}\left(\frac{u^2}{\ell}\cdot\frac{u}{\ell^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda^3}{\ell^3}\right), \tag{3.3.54}$$

$$\Omega_j\,\overline{\omega_i s_{ij}} = \mathcal{O}\left(\frac{u}{\ell}\cdot\frac{u^2}{\ell^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda^3}{\ell^3}\right), \tag{3.3.55}$$

$$\nu\frac{\partial^2}{\partial x_j\,\partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) = \mathcal{O}\left(\frac{\nu}{\ell^2}\cdot\frac{u^2}{\lambda^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda^3}{\ell^3}\right), \tag{3.3.56}$$

$$\overline{\omega_i\omega_j}\,S_{ij} = \mathcal{O}\left(\frac{u^2}{\lambda\ell}\cdot\frac{u^2}{\ell}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda^2}{\ell^2}\right), \tag{3.3.57}$$

$$\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j\omega_i\omega_i}\right) = \mathcal{O}\left(\frac{u}{\ell}\cdot\frac{u^2}{\lambda^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda}{\ell}\right), \tag{3.3.58}$$

$$U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}\,\overline{\omega_i\omega_i}\right) = \mathcal{O}\left(\frac{u}{\ell}\cdot\frac{u^2}{\lambda^2}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot\frac{\lambda}{\ell}\right), \tag{3.3.59}$$

$$\overline{\omega_i\omega_j s_{ij}} = \mathcal{O}\left(\frac{u^2}{\lambda^2}\cdot\frac{u}{\lambda}\right) = \mathcal{O}\left(\frac{u^3}{\lambda^3}\cdot 1\right), \tag{3.3.60}$$

$$\nu\overline{\frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_i}{\partial x_j}} = ? \tag{3.3.61}$$

In the stretching term (3.3.60), no prorating with λ/ℓ is necessary, because ω_i operates on the same time scale as s_{ij}. The viscous dissipation term (3.3.61) has been left undecided, since we expect dissipation of vorticity to occur mainly at length scales smaller than λ. In the viscous diffusion term (3.3.56), the relation $\ell^2/\lambda^2 \sim u\ell/\nu$ has been used. In the transport term (3.3.59), the operator $U_j\partial/\partial x_j$ has been estimated as u/ℓ; that choice is consistent with the estimates used in the equations for the mean flow and the turbulent kinetic energy (see 3.2.28, 3.2.31, 3.2.32).

The expressions (3.3.54) through (3.3.60) have been arranged in increasing order of magnitude. If the Reynolds number is large, all of the terms (3.3.54) through (3.3.59) are smaller than the turbulent stretching term (3.3.60) by at least a factor of λ/ℓ, which is of order $R_\ell^{-1/2}$. Therefore, at sufficiently high

Reynolds numbers the turbulent vorticity budget (3.3.38) may be approximated as (Taylor, 1938)

$$\overline{\omega_i \omega_j s_{ij}} = \nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}} . \tag{3.3.62}$$

The budget of mean-square vorticity fluctuations is thus approximately independent of the structure of the mean flow. Turbulent vorticity fluctuations, unlike turbulent velocity fluctuations, do not need the continued presence of a source term associated with the mean flow field. Of course, in the absence of a source of energy, turbulent vorticity fluctuations will decay, too. Also, the rate of change of $\overline{\omega_i \omega_i}$, as represented by (3.3.59), is small compared to the rate at which turbulent vortex stretching occurs. In Chapter 8 it will be shown that these conclusions lead to the concept of an equilibrium spectrum of turbulence at small scales.

The right-hand side of (3.3.62) is quadratic in $\partial \omega_i / \partial x_j$, so that it is always *positive*. Hence, the left-hand side is positive, too. This implies that, on the average, there is more turbulent vortex stretching than vortex squeezing: vortex stretching transfers turbulent vorticity (and the energy associated with it) from large-scale fluctuations to small-scale fluctuations. In this way turbulence obtains the broad energy spectrum that is observed experimentally, and in this way the very smallest eddies (which suffer rapid viscous decay) are continually being supplied with new energy. The approximate vorticity budget (3.3.62) is just as essential to understanding turbulence dynamics as the approximate energy budget (3.2.6). The relationship between these two budgets, incidentally, is a close one: viscous dissipation of vorticity prevents vorticity production ($\overline{\omega_i \omega_j s_{ij}}$) from increasing $\overline{\omega_i \omega_i}$ without limit, while viscous dissipation of energy (which is proportional to $\overline{\omega_i \omega_i}$) prevents the energy production ($-\overline{u_i u_j} S_{ij}$) from increasing $\overline{u_i u_i}$ without limit. Vortex stretching makes $\overline{\omega_i \omega_i}$ as large as viscosity will permit; at large Reynolds numbers the mean-square strain-rate fluctuations keep pace, so that the turbulent energy is subject to rapid dissipation.

Two points need to be emphasized. First, in two-dimensional "turbulence" there is no vortex stretching, so that the vorticity budget (3.3.62) is irrelevant in that case. This implies that the spectral energy-transfer concepts developed here do not apply to two-dimensional stochastic flow fields.

Second, vorticity amplification is a result of the kinematics of turbulence. As an example, take a situation in which the principal axes of the instantaneous strain rate are aligned with the coordinate system, so that s_{ij} has only diagonal components (s_{11}, s_{22}, and s_{33}). Let us assume for simplicity that $s_{22} = s_{33}$, so that, by virtue of continuity, $s_{11} = -2s_{22}$. The term $\omega_i \omega_j s_{ij}$ becomes, if we also assume that $\omega_2^2 \cong \omega_3^2$,

$$\omega_1^2 s_{11} + \omega_2^2 s_{22} + \omega_3^2 s_{33} = s_{11}(\omega_1^2 - \omega_2^2). \tag{3.3.63}$$

If $s_{11} > 0$, ω_1^2 is amplified (see Figure 3.4), but ω_2^2 and ω_3^2 are attenuated because s_{22} and s_{33} are negative. Thus, $\omega_1^2 - \omega_2^2$ tends to become positive if s_{11} is positive. Again, if $s_{11} < 0$, ω_1^2 decreases, but ω_2^2 and ω_3^2 increase, so that $\omega_1^2 - \omega_2^2 < 0$, making the stretching term positive again.

Multiple length scales If the vorticity gradients $\partial\omega_i/\partial x_j$ in (3.3.62) were estimated as u/λ^2, the dissipation term would be smaller than the stretching term. However, λ is not the proper length scale for estimates of ω_i and u is not the proper velocity scale; all we know is that the ratio u/λ is the order of magnitude of ω_i. Clearly, we need a new length scale. Calling it δ, using (3.3.60), and requiring that the two sides of (3.3.62) have the same order of magnitude, we obtain

$$\nu \frac{u^2}{\lambda^2 \delta^2} = \mathcal{O}\left(\frac{u^3}{\lambda^3}\right) . \tag{3.3.64}$$

The ratio δ/λ becomes

$$\delta/\lambda = \mathcal{O}(\nu/u\lambda)^{1/2} = \mathcal{O}(R_\lambda^{-1/2}). \tag{3.3.65}$$

Comparing this with (3.2.18), we see that δ is proportional to the Kolmogorov microscale η. The Kolmogorov microscale thus has a role in the turbulent vorticity budget which is comparable to the role of the Taylor microscale in the turbulent energy budget. Since vortex stretching is the only known spectral energy-transfer mechanism, η is the smallest length scale possible: the dynamics of $\overline{(\partial\omega_i/\partial x_j)^2}$ would not lead to a length scale smaller than η.

Since the vorticity budget is approximately independent of the structure of the mean flow, vorticity dynamics can be studied more easily in the wave-number (spectral) domain than in the spatial domain. This subject, therefore, is taken up again in Chapter 8.

Stretching of magnetic field lines The dynamics of the fluctuating vorticity is representative of the dynamics of other axial vector fields in turbulent flow. For example, magnetic field lines in a conducting fluid are stretched by fluctuating strain rates much like vortex lines. In incompressible fluids with constant properties, charge equilibrium, negligible displacement currents and radiation, the equation for the magnetic field is the same as the equation for vorticity. If the magnetic energy is small compared to the kinetic energy, the magnetic field is a passive contaminant which does not change the velocity field appreciably. In that case magnetic-field fluctuations are intensified only by fluctuating strain rates, and an approximate equation for the fluctuations h_i of the magnetic field reads, in analogy with (3.3.62) (Saffman, 1963),

$$\overline{h_i h_j s_{ij}} = \gamma_m \overline{\frac{\partial h_j}{\partial x_i} \frac{\partial h_i}{\partial x_j}} . \qquad (3.3.66)$$

This equation states that the amplification of $\overline{h_i h_i}$ by strain-rate fluctuations is kept in balance by ohmic dissipation of $\overline{h_i h_i}$ (the right-hand side of (3.3.66) is proportional to j^2/σ, where j is the current density and σ is the electrical conductivity).

If the magnetic diffusivity γ_m differs from ν, the dissipative length scale of the magnetic-field fluctuations is different from the Kolmogorov microscale η. If the dissipative length scale for h_i is called η_m and if the rms value of h_i is called $\mathscr{h}$, we may estimate (3.3.66) by

$$\mathscr{h}^2 \cdot u/\lambda \sim \gamma_m \mathscr{h}^2/\eta_m^2 . \qquad (3.3.67)$$

Because the magnetic-field fluctuations are generated by fluctuating strain rates, the correlation coefficient between $h_i h_j$ and s_{ij} should be of order one. Because we are interested only in estimates for scales, we ignore all numerical factors that are of order one. Using the scale relation $u/\lambda \sim (\epsilon/\nu)^{1/2}$ and the definition of η ($\eta = (\nu^3/\epsilon)^{1/4}$), and absorbing numerical coefficients in the definition of η_m, we obtain

$$\eta_m/\eta = (\gamma_m/\nu)^{1/2} . \qquad (3.3.68)$$

If the fluid is a very good conductor of electricity so that $\gamma_m/\nu \ll 1$, this implies that the spectrum of $\overline{h_i h_i}$ extends to scales much smaller than η. The possibility of achieving scales smaller than η, even though h_i is a passive contaminant, arises because the strain rate stretches the magnetic field into thin filaments if the magnetic diffusivity is small. The scale-reducing effect of the strain rate proceeds until it is checked by the magnetic diffusivity (see Figure 3.6). This effect is similar to that observed in mixing paint of different colors. The diffusivity of pigment is quite small relative to the kinematic viscosity of paint; it takes long, patient stirring before the filaments of different color have become so thin and so close together that the molecular diffusivity of pigment can homogenize the mixture.

In interstellar gas clouds consisting mainly of ionized hydrogen, γ_m/ν may be as small as 10^{-8}, so that the smallest magnetic eddies are quite small compared to η. In liquid metals and electrolytes, on the other hand, $\gamma_m/\nu >> 1$, so that the smallest magnetic eddies are large compared to η. If this is the case, the estimate $s_{ij} \sim (\epsilon/\nu)^{1/2}$ has to be revised, because the strain rate at scales comparable to the magnetic microscale η_m is smaller than $(\epsilon/\nu)^{1/2}$ if $\eta_m >> \eta$. In other words, the viscosity cannot be used as a scaling parameter at scales large compared to η. The only alternative is to construct a strain rate from ϵ and η_m; this yields $s_{ij} \sim \epsilon^{1/3}\, \eta_m^{-2/3}$ (see also Section 8.6). If we use this instead of u/λ in (3.3.67), we obtain

$$\eta_m = (\gamma_m^3/\epsilon)^{1/4}, \quad \eta_m/\eta = (\gamma_m/\nu)^{3/4}. \tag{3.3.69}$$

A note of warning is in order, because there may be no magnetic eddies at all if γ_m/ν is large enough. In mercury, $\gamma_m/\nu = 7 \times 10^6$, so that the *magnetic Reynolds number* $R_m = u\ell/\gamma_m$ is less than one if $R = u\ell/\nu < 7 \times 10^6$. If $R_m < 1$, the generation of magnetic-field fluctuations is prevented by the magnetic diffusivity, much as turbulent motion cannot exist if $R < 1$. In that case,

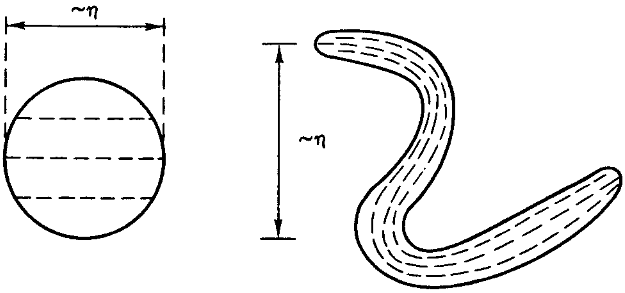


Figure 3.6. A magnetic eddy (a) of scale η is stretched by the strain rate into a thin filament (b). If $\gamma_m << \nu$, the gradients in magnetic field intensity can become quite steep (the dashed lines represent surfaces of constant h).

there can be only a mean magnetic field, which affects the velocity turbulence if it is strong enough.

3.4 The dynamics of temperature fluctuations

The equations governing turbulent fluctuations of vectors (such as vorticity) are complicated because vectors interact with a flow field in a variety of ways. However, scalar contaminants (such as temperature) are governed by fairly simple equations, as we have seen in Chapter 2. We shall discuss the dynamics of temperature fluctuations in an incompressible turbulent flow as an example of the dynamics of all other passive scalar contaminants.

The equation governing the dynamics of $\overline{\theta^2}$ in a steady flow is obtained in exactly the same way as the equations for $\overline{u_i u_i}$ and $\overline{\omega_i \omega_i}$. The result is

$$U_j \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{\theta^2}) = -\frac{\partial}{\partial x_j}\left[\tfrac{1}{2}\overline{\theta^2 u_j} - \gamma \frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{\theta^2})\right] - \overline{\theta u_j}\frac{\partial \Theta}{\partial x_j} - \gamma \overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}}. \quad (3.4.1)$$

The rate of change of $\overline{\theta^2}$ is thus controlled by turbulent and molecular transport of $\overline{\theta^2}$ (the first two terms on the right-hand side of the equation), by gradient production (which is like the production term of turbulent kinetic energy), and by molecular dissipation (γ is the thermal diffusivity). In a steady homogeneous shear flow, (3.4.1) reduces to

$$-\overline{\theta u_j}\frac{\partial \Theta}{\partial x_j} = \gamma \overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}}, \quad (3.4.2)$$

which states that gradient production of $\overline{\theta^2}$ is balanced by the molecular "smearing" of temperature fluctuations.

If there is only one temperature scale and one length scale, $\overline{\theta u_j}$ is of order $\theta' u$ and $\partial \Theta / \partial x_j$ is of order θ'/ℓ (θ' is the rms temperature fluctuation). The left-hand side of (3.4.2) is then of order $\overline{\theta^2} u/\ell$, which is consistent with the idea that spectral transfer of temperature fluctuations toward the dissipative range of eddy sizes should proceed at a rate dictated by the characteristic time of large eddies (ℓ/u) and the amount of $\overline{\theta^2}$ that is involved.

Microscales in the temperature field The right-hand side of (3.4.2) requires the introduction of a Taylor microscale for the temperature fluctuations. Let us define

$$\overline{(\partial \theta / \partial x_1)^2} \equiv 2\, \overline{\theta^2} / \lambda_\theta^2 . \quad (3.4.3)$$

The coefficient 2 in (3.4.3) is a normalization factor, which brings (3.4.3) into agreement with the expressions used in the literature (see also Chapter 6). If the small-scale structure of the temperature field is isotropic, $\overline{(\partial\theta/\partial x_1)^2} = \overline{(\partial\theta/\partial x_2)^2} = \overline{(\partial\theta/\partial x_3)^2}$, so that the right-hand side of (3.4.2) becomes

$$\gamma \overline{\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_j}} = 6\gamma \frac{\overline{\theta^2}}{\lambda_\theta^2}. \tag{3.4.4}$$

An estimate for λ_θ is obtained by requiring that both sides of (3.4.2) have the same order of magnitude. Recalling that $-\overline{\theta u_j}\,\partial\Theta/\partial x_j \sim \theta^2 u/\ell$ (as discussed previously) and that $(\ell/\lambda)^2 \sim u\ell/\nu$, we find

$$\lambda_\theta/\lambda = C(\gamma/\nu)^{1/2}. \tag{3.4.5}$$

The constant C is of order one (Corrsin, 1951).

The Taylor microscale for temperature, λ_θ, is an artificial length scale, just like λ. If we want to determine the dissipative eddy size of the temperature field, we have to consult the equation governing temperature gradients. In analogy with (3.3.62) and (3.3.66), the equation for $\overline{(\partial\theta/\partial x_i)(\partial\theta/\partial x_j)}$ may be approximated by (Corrsin, 1953)

$$\overline{\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}}\, s_{ij} = \gamma \overline{\frac{\partial^2\theta}{\partial x_i \partial x_j}\frac{\partial^2\theta}{\partial x_i \partial x_j}}. \tag{3.4.6}$$

If $\gamma < \nu$, most of the dissipation of temperature-gradient fluctuations occurs at scales smaller than η, so that the temperature field is exposed to the entire spectrum of strain-rate fluctuations. Consequently, the proper estimate for s_{ij} is $(\epsilon/\nu)^{1/2}$ in this case. In analogy with (3.3.68), the temperature microscale η_θ is then given by (Batchelor, 1959; see also Section 8.6)

$$\eta_\theta/\eta = (\gamma/\nu)^{1/2}. \tag{3.4.7}$$

If the thermal diffusivity γ and the kinematic viscosity ν are approximately equal (as in gases), temperature fluctuations extend to scales as small as η. In liquids, the microscales may be different. For water, the Prandtl number ν/γ is about 7, so that temperature fluctuations extend to scales almost 3 times as small as η. The creation of very small temperature eddies in a fluid with a large Prandtl number is due to the straining effect illustrated in Figure 3.6.

If $\gamma > \nu$, so that the Prandtl number is smaller than one, η_θ is larger than η. In this case, even the very smallest temperature eddies are not exposed to the entire spectrum of strain-rate fluctuations. If $\gamma >> \nu$, the effective value of the strain rate must be independent of ν. This leads to $s_{ij} \sim \epsilon^{1/3}\, \eta_\theta^{-2/3}$; in analogy with (3.3.69), the temperature microscale becomes (Oboukhov, 1949; Corrsin, 1951; see also Section 8.6)

$$\eta_\theta = (\gamma^3/\epsilon)^{1/4}, \quad \eta_\theta/\eta = (\gamma/\nu)^{3/4}. \tag{3.4.8}$$

This estimate applies to liquid metals and electrolytes, in which the Prandtl number is small (for mercury, $\nu/\gamma = 0.028$).

Buoyant convection One interesting group of problems arises when temperature is not a passive but an active contaminant which can contribute to the generation of velocity fluctuations. The case we have in mind is thermal convection in gases exposed to a gravity field. Temperature fluctuations cause density fluctuations in a gas at essentially constant pressure (that is, very low Mach number). The density fluctuations cause a fluctuating body force $g_i\rho'/\overline{\rho}$ (g_i is the vector acceleration of gravity, ρ' is the density fluctuation, and ρ is the mean density). In the *Boussinesq approximation*, the fluctuating body force is written as $-g_i\vartheta/\Theta_0$, where Θ_0 is the mean temperature of an adiabatic atmosphere and ϑ is the difference between the actual temperature and Θ_0. The adiabatic temperature Θ_0 changes in the direction of the gravity vector in response to the gravity-induced pressure gradient, but the length scale involved is large, so that Θ_0 may be treated as a constant in many problems (Lumley and Panofsky, 1964).

The temperature difference ϑ is decomposed into a mean value $\overline{\vartheta}$ and fluctuations $\theta\,(\overline{\theta} = 0)$. If $U_i = 0$, the fluctuating body force performs work at a mean rate $-g_i\,\overline{\theta u_i}/\Theta_0$. This work, called *buoyant production*, must be added as a source term in the budget of turbulent kinetic energy. The heat flux $c_p\rho\overline{\theta u_i}$ then assumes a dual role, because it occurs in production terms for both $\frac{1}{2}\overline{u_iu_i}$ and $\overline{\theta^2}$.

In a flow that is steady and homogeneous in the x_1, x_2 plane and in which the only nonzero components of U_i and g_i are $U_1 = U_1\ (x_3)$ and $g_3 = -g$ (it is consistent with geophysical practice to take the x_3 direction vertically upwards), the heat and momentum fluxes $\rho c_p\,\overline{\theta u_3}$ and $\rho\overline{u_1u_3}$ are constant if molecular transport of $\overline{\vartheta}$ and U_1 in the x_3 direction can be neglected. The equations for $\frac{1}{2}\overline{u_iu_i}$ and $\overline{\theta^2}$ reduce to

$$0 = -\overline{u_1u_3}\frac{\partial U_1}{\partial x_3} + \frac{g}{\Theta_0}\overline{u_3\theta} - \frac{\partial}{\partial x_3}\left(\frac{1}{2}\overline{u_iu_iu_3} + \frac{1}{\rho}\overline{pu_3}\right) - \nu\,\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}, \tag{3.4.9}$$

$$0 = -\overline{\theta u_3}\frac{\partial \overline{\vartheta}}{\partial x_3} - \frac{\partial}{\partial x_3}(\tfrac{1}{2}\overline{\theta^2 u_3}) - \gamma \overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}}. \tag{3.4.10}$$

In these equations the terms representing transport of kinetic energy and temperature variance by molecular motion have been neglected because they are ordinarily very small. The mean temperature gradient $\partial\overline{\vartheta}/\partial x_3$ is equal to the actual temperature gradient minus the gravity-induced temperature gradient $\partial\Theta_0/\partial x_3 = -g/c_p$ which would exist in a flow without heat transfer (the set $\partial\Theta_0/\partial x_3 = -g/c_p$, $(1/\rho_0)\ \partial P_0/\partial x_3 = -g$, $P_0/\rho_0 = R\Theta_0$ defines a perfect-gas atmosphere in which the entropy is constant).

The two equations (3.4.9, 3.4.10) are used in the study of atmospheric turbulence. The outstanding feature of these equations, of course, is the buoyant production of kinetic energy. Apparently, there exist situations in which turbulence need not be maintained by shear stresses because it can be maintained by fluctuating buoyancy forces. Turbulence driven by body forces is not nearly as well understood as turbulence driven by shear stresses; for example, no satisfactory theory of atmospheric turbulence in unstable conditions ($\partial\overline{\vartheta}/\partial x_3 < 0$) exists.

Richardson numbers Some of the parameters governing (3.4.9, 3.4.10) need to be introduced. The most obvious one is the ratio of buoyant production to stress production of turbulent kinetic energy. This parameter is called the *flux Richardson number*; it is defined as

$$R_f \equiv \frac{g}{\Theta_0}\frac{\overline{u_3\theta}}{\overline{u_1u_3}\ \partial U_1/\partial x_3}. \tag{3.4.11}$$

If the heat transfer is upward ($\overline{u_3\theta} > 0$), the value of R_f is negative because $\overline{u_1u_3} < 0$ if $\partial U_1/\partial x_3 > 0$. As (3.4.9) indicates, the production of turbulent kinetic energy is increased in this case. Upward heat flux generally corresponds to $\partial\overline{\vartheta}/\partial x_3 < 0$; this is called an *unstable* atmosphere. If the heat transfer is downward ($\overline{\theta u_3} < 0$), $R_f > 0$, and the buoyant-production term becomes negative, indicating that kinetic energy is lost. Negative values of $\overline{\theta u_3}$ generally correspond to positive values of $\partial\overline{\vartheta}/\partial x_3$; this is called *stable* stratification. If a positive R_f becomes large enough, it leads to complete suppression of all turbulence.

If we define an eddy viscosity and an eddy conductivity by

$$-\overline{u_1 u_3} \equiv \nu_T \partial U_1/\partial x_3, \tag{3.4.12}$$

$$-\overline{\theta u_3} \equiv \gamma_T \partial \overline{\vartheta}/\partial x_3, \tag{3.4.13}$$

the flux Richardson number may be written as

$$R_f = \frac{\gamma_T g}{\nu_T \Theta_0} \frac{\partial \overline{\vartheta}/\partial x_3}{(\partial U_1/\partial x_3)^2}. \tag{3.4.14}$$

Apart from the "exchange" coefficients ν_T and γ_T, this expression contains variables that can be measured with relative ease. This suggests that a different parameter, the *gradient Richardson number*, should be useful:

$$R_g \equiv \frac{g}{\Theta_0} \frac{\partial \overline{\vartheta}/\partial x_3}{(\partial U_1/\partial x_3)^2}. \tag{3.4.15}$$

If ν_T and γ_T are approximately the same (which may be a very unreliable assumption if the absolute value of R_f is not small), the parameters R_f and R_g are approximately the same, too. Observations have shown that turbulence cannot be maintained if $R_f > 0.2$ approximately.

Buoyancy time scale The group $(g/\Theta_0)\partial\overline{\vartheta}/\partial x_3$ in (3.4.15) has dimensions $\sec^{-2}$. If $\partial\overline{\vartheta}/\partial x_3 > 0$ (stable conditions), we define

$$(g/\Theta_0)\partial\overline{\vartheta}/\partial x_3 \equiv N_b^2; \tag{3.4.16}$$

if $\partial\vartheta/\partial x_3 < 0$ (unstable conditions), we define

$$-(g/\Theta_0)\partial\overline{\vartheta}/\partial x_3 \equiv T_b^{-2}. \tag{3.4.17}$$

The parameter N_b is called the Brunt-Väisälä frequency; it is the frequency of gravity waves in a stable atmosphere. In an unstable atmosphere, gravity waves are unstable and break up into turbulence. Therefore, if $\partial\overline{\vartheta}/\partial x_3 < 0$ we use the buoyancy time scale T_b. In sunny weather, T_b is typically of the order of a few minutes; more strongly unstable conditions correspond to smaller values of T_b. In a neutral atmosphere ($\partial\overline{\vartheta}/\partial x_3 = 0$), the time scale $T_b \to \infty$, and the frequency $N_b = 0$.

The mean wind gradient $\partial U_1/\partial x_3$ has the dimensions $\sec^{-1}$. If we define

$$\partial U_1/\partial x_3 \equiv T_s^{-1}, \tag{3.4.18}$$

we obtain

$$R_g = (N_b T_s)^2, \quad (\partial\bar{\vartheta}/\partial x_3 > 0), \tag{3.4.19}$$

$$R_g = -(T_s/T_b)^2, \quad (\partial\bar{\vartheta}/\partial x_3 < 0). \tag{3.4.20}$$

We conclude that the gradient Richardson number is the square of a ratio of time scales.

Monin-Oboukhov length In the surface layer of the atmosphere (which may extend up to several tens of meters above the surface), different parameters are important, so that the Richardson number is arranged in a different way. We assume that the wind profile is logarithmic: $\partial U_1/\partial x_3 = u_*/\kappa x_3$ (see Section 2.3). The Reynolds stress $-\rho\,\overline{u_1 u_3}$ is constant; it is put equal to ρu_*^2 (u_* is the friction velocity). The flux Richardson number then reads

$$R_f = -\frac{\kappa g x_3 \overline{\theta u_3}}{\Theta_0 u_*^3}. \tag{3.4.21}$$

The heat flux $H = \rho c_p\, \overline{\theta u_3}$; if we define a length L by

$$L \equiv -\frac{\Theta_0 u_*^3}{\kappa g \overline{\theta u_3}} = -\frac{c_p \rho \Theta_0 u_*^3}{\kappa g H}, \tag{3.4.22}$$

we obtain

$$R_f = x_3/L. \tag{3.4.23}$$

The length L is known as the *Monin-Oboukhov length scale.* Monin and Oboukhov have successfully used x_3/L as the basic independent variable for the description of the surface layer, both in stable and unstable conditions. The absolute value of L is seldom less than 10 m, so that the conditions in the lowest meter of the atmosphere are approximately neutral, except when the wind speed is very low.

Convection in the atmospheric boundary layer As an illustration of the complexity of the problems caused by buoyant production of turbulence, let us consider atmospheric boundary layers in unstable conditions ($\partial\bar{\vartheta}/\partial x_3 < 0$). In the surface layer of these boundary layers the absolute value of R_f is small, but at heights above 50 m, say, we may expect production by Reynolds stresses to be very small compared to buoyant production if the upward heat flux is appreciable (sunny afternoon weather). Also, the turbulence outside the surface layer is thoroughly mixed by the thermal convection, so that transport terms in the energy budget (3.4.9) should be small. An approximate energy budget for the turbulence above the surface layer then reads

$$\frac{g}{\Theta_0}\overline{\theta u_3} \cong \nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}. \tag{3.4.24}$$

Let us assume that θ and u_3 are well correlated, so that $\overline{\theta u_3} \sim tw$ if the rms values of θ and u_3 are represented by t and w. In turbulence with velocity scale w and length scale h, the dissipation rate is of order w^3/h (h scales with the height of the atmospheric boundary layer). Substituting the estimates $\overline{\theta u_3} \sim tw$ and $\epsilon \sim w^3/h$ into (3.4.24), we obtain

$$w^2 \sim g\,t\,h/\Theta_0. \tag{3.4.25}$$

This estimate states that a buoyant acceleration of order gt/Θ_0, acting over a distance h, produces kinetic energy of order gth/Θ_0.

If the heat flux $\rho c_p \overline{\theta u_3}$ throughout the boundary layer is of the same order as the heat flux in the surface layer, $\overline{\theta u_3}$ can be written in terms of the Monin-Oboukhov length L defined in (3.4.22) (note that the Monin-Oboukhov length is defined on basis of the surface heat flux). This yields

$$\overline{\theta u_3} \sim wt \sim -\Theta_0 u_*{}^3/gL. \tag{3.4.26}$$

Substituting for t with (3.4.26) in (3.4.25), we obtain

$$(w/u_*)^2 \sim (-h/L)^{2/3}. \tag{3.4.27}$$

As the heat flux increases, the value of $-L$ ($L < 0$ if $\overline{\theta u_3} > 0$) decreases. A value of $-L$ representative of strong convection is $-L = 10$ m; the height h is of the order of 1,000 m. We conclude from (3.4.27) that the kinetic energy $\frac{1}{2}w^2$ of the turbulence above the surface layer becomes large compared to u_*^2 if the upward heat flux is large (in the absence of heat transfer, $w \sim u_*$). This implies that the correlation between u_1 and u_3 is small under these conditions, because $u_1 \sim w$, $u_3 \sim w$, but $\overline{u_1 u_3} \sim u_*^2$. Turbulent eddies created by buoyancy forces apparently cause relatively little momentum transfer. This undermines the foundation on which eddy-viscosity and mixing-length expressions are based, so that they cannot be used in a complicated problem like this.

In a flow with temperature fluctuations of order t and with a length scale

h, the mean temperature gradient $\partial\overline{\vartheta}/\partial x_2$ is at most of order t/h if the thermal convection keeps the temperature field mixed. Thus, the buoyancy time scale T_b defined in (3.4.17) may be estimated as

$$T_b \sim (gt/\Theta_0 h)^{-1/2}. \qquad (3.4.28)$$

Substituting for t with (3.4.26, 3.4.27), we obtain

$$T_b \sim (h/u_*)(-L/h)^{1/3}. \qquad (3.4.29)$$

The height h of the boundary layer often is of order u_*/f, where f is the Coriolis parameter (Blackadar and Tennekes, 1968). If this is the case, (3.4.29) becomes

$$T_b f \sim (-L/h)^{1/3}. \qquad (3.4.30)$$

Clearly, the problem of buoyant convection is one with two time scales, that is, T_b and f^{-1}, which may differ by an order of magnitude if $-L$ and h differ by a few orders of magnitude. As we have seen before, most problems in turbulence theory that involve more than one dynamically significant time or length scale are so complicated that no comprehensive solution is possible at the present state of the art.

Buoyancy-generated eddies cause relatively little momentum transport, but they are quite effective in transporting heat. In other words, the ratio of the turbulent diffusivities for heat and momentum is much larger than one, so that Reynolds' analogy (Section 2.4) does not apply.

Problems

3.1 Estimate the characteristic velocity of eddies whose size is equal to the Taylor microscale λ (see Problem 1.3). Use this estimate to show that eddies of this size contribute very little to the total dissipation rate.

3.2 Experimental evidence suggests that the dissipation rate is not evenly distributed over the volume occupied by a turbulent flow. The distribution of the dissipation rate appears to be intermittent, with large dissipation rates occupying a small volume fraction. Make a model of this phenomenon by assuming that all of the dissipation occurs in thin vortex tubes (diameter η, characteristic velocity $u = [\frac{1}{3}\overline{u_i u_i}]^{1/2}$). What is the volume fraction occupied by these tubes? Verify if the approximate vorticity budget (3.3.62) indeed holds for these vortex tubes.

3.3 A qualitative estimate of the effect of a wind-tunnel contraction (Figure 3.5) on turbulent motion can be obtained by assuming that the angular momentum of eddies does not change through the contraction. Let the contraction ratio, which is equal to the ratio of the mean velocity behind the contraction to that in front of the contraction, be equal to c. Show that the velocity fluctuations associated with an "eddy" aligned with the mean flow (as in Figure 3.5) increase by a factor $c^{1/2}$ and that those associated with an eddy perpendicular to the mean flow decrease by a factor c. Compute the effect of the contraction on the relative turbulence intensity u/U. Estimate the effect of the contraction on the rate of decay of velocity fluctuations. Is it feasible to design a contraction such that the evolution of turbulent velocity fluctuations during the contraction can be ignored?

3.4 A fully developed turbulent pipe flow of fluid with a Prandtl number equal to one is being cooled by the addition of a small volume of slightly cooler fluid over a cross section. Estimate the initial temperature fluctuation level. How many pipe diameters downstream are required before the temperature fluctuations have decayed to 1% of the initial level? For the purpose of this calculation, it may be assumed that the mean velocity in the pipe is approximately independent of position. Also, an estimate for the dissipation rate ϵ is needed; it can be obtained from momentum and energy integrals for pipe flow. For a prescribed decrease in mean temperature in the pipe, should one increase the volume flow of coolant and reduce the temperature difference or vice versa in order to reduce the temperature fluctuations?