2

TURBULENT TRANSPORT OF MOMENTUM AND HEAT

Turbulence consists of random velocity fluctuations, so that it must be treated with statistical methods. The statistical analysis does not need to be sophisticated at this stage; a simple decomposition of all quantities into mean values and fluctuations with zero mean will suffice for the next few chapters. We shall find that turbulent velocity fluctuations can generate large momentum fluxes between different parts of a flow. A momentum flux can be thought of as a stress; turbulent momentum fluxes are commonly called Reynolds stresses. The momentum exchange mechanism superficially resembles molecular transport of momentum. The latter gives rise to the viscosity of a fluid; by analogy, the turbulent momentum exchange is often represented by an eddy viscosity. This analogy will be explored in great detail.

2.1
The Reynolds equations

In turbulence, a description of the flow at all points in time and space is not feasible. Instead, following Reynolds (1895), we develop equations governing mean quantities, such as the mean velocity. The equations of motion of an incompressible fluid are

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij}, \tag{2.1.1}$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0. \tag{2.1.2}$$

Here, $\tilde{\sigma}_{ij}$ is the *stress tensor*. Repeated indices in any term indicate a summation over all three values of the index; a tilde denotes the instantaneous value at (x_i, t) of a variable on which no Reynolds decomposition into a mean value and fluctuations (see next section) has been performed.

If the fluid is Newtonian, the stress tensor $\tilde{\sigma}_{ij}$ is given by

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + 2\mu\tilde{s}_{ij}. \tag{2.1.3}$$

In (2.1.3), δ_{ij} is the Kronecker delta, which is equal to one if $i = j$ and zero otherwise; $\tilde{p}$ is the hydrodynamic pressure and μ is the dynamic viscosity (which will be assumed to be constant). The *rate of strain* $\tilde{s}_{ij}$ is defined by

$$\tilde{s}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right). \tag{2.1.4}$$

If (2.1.3) is substituted into (2.1.1) and if the continuity equation (2.1.2) is invoked, the *Navier-Stokes equations* are obtained:

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \quad . \tag{2.1.5}$$

Here, ν is the kinematic viscosity ($\nu = \mu/\rho$).

The Reynolds decomposition The velocity $\tilde{u}_i$ is decomposed into a mean flow U_i and velocity fluctuations u_i, such that

$$\tilde{u}_i = U_i + u_i \, . \tag{2.1.6}$$

We interpret U_i as a time average, defined by

$$U_i = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \tilde{u}_i \, dt. \tag{2.1.7}$$

Time averages (mean values) of fluctuations (which are denoted by lowercase letters) and of their derivatives, products, and other combinations are denoted by an overbar. The mean value of a fluctuating quantity itself is zero by definition; for example,

$$\overline{u_i} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} (\tilde{u}_i - U_i) \, dt \equiv 0. \tag{2.1.8}$$

The use of time averages corresponds to the typical laboratory situation, in which measurements are taken at fixed locations in a statistically steady, but often inhomogeneous, flow field. In an inhomogeneous flow, a time average like U_i is a function of position, so that the use of a spatial average would be inappropriate for most purposes. For a time average to make sense, the integrals in (2.1.7) and (2.1.8) have to be independent of t_0. In other words, the mean flow has to be steady:

$$\frac{\partial U_i}{\partial t} = 0. \tag{2.1.9}$$

Without this constraint (2.1.7) and (2.1.8) would be meaningless. The averaging time T needed to measure mean values depends on the accuracy desired; this problem is discussed in Section 6.4.

The mean value of a spatial derivative of a variable is equal to the corresponding spatial derivative of the mean value of that variable; for example,

$$\frac{\partial \overline{\tilde{u}_i}}{\partial x_j} = \frac{\partial U_i}{\partial x_j}, \quad \frac{\overline{\partial u_i}}{\partial x_j} = \frac{\partial}{\partial x_j} \overline{u_i} = 0. \tag{2.1.10}$$

These operations can be performed because averaging is carried out by integrating over a long period of time, which commutes with differentiation with respect to another independent variable.

The pressure $\tilde{p}$ and the stress $\tilde{\sigma}_{ij}$ are also decomposed into mean and fluctuating components. Again, capital letters are used for mean values and lowercase letters for fluctuations with zero mean. Specifically,

$$\tilde{p} = P + p, \quad \overline{p} \equiv 0, \tag{2.1.11}$$

$$\tilde{\sigma}_{ij} = \Sigma_{ij} + \sigma_{ij}, \quad \overline{\sigma}_{ij} \equiv 0. \tag{2.1.12}$$

Like U_i, P and Σ_{ij} are independent of time. The mean stress tensor Σ_{ij} is given by

$$\Sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij}, \tag{2.1.13}$$

and the stress fluctuations σ_{ij} are given by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu s_{ij}. \tag{2.1.14}$$

Here, the mean strain rate S_{ij} and the strain-rate fluctuations s_{ij} are defined by

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad s_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.1.15}$$

The commutation between averaging and spatial differentiation involved here is based on (2.1.10).

Correlated variables Averages of products are computed in the following way:

$$\begin{aligned} \overline{\tilde{u}_i \tilde{u}_j} &= \overline{(U_i + u_i)(U_j + u_j)} \\ &= U_i U_j + \overline{u_i u_j} + \overline{U_i u_j} + \overline{U_j u_i} \\ &= U_i U_j + \overline{u_i u_j}. \end{aligned} \tag{2.1.16}$$

The terms consisting of a product of a mean value and a fluctuation vanish if they are averaged, because the mean value is a mere coefficient as far as the averaging is concerned, and the average of a fluctuating quantity is zero.

If $\overline{u_i u_j} \neq 0$, u_i and u_j are said to be *correlated*; if $\overline{u_i u_j} = 0$, the two are *uncorrelated*. Figure 2.1 illustrates the concept of correlated fluctuating variables. A measure for the degree of correlation between the two variables u_i and u_j is obtained by dividing $\overline{u_i u_j}$ by the square root of the product of the *variances* $\overline{u_i^2}$ and $\overline{u_j^2}$; this gives a *correlation coefficient* c_{ij}, which is defined by

$$c_{ij} \equiv \overline{u_i u_j} / (\overline{u_i^2} \cdot \overline{u_j^2})^{1/2}, \tag{2.1.17}$$

with the understanding that the summation convention does not apply in this case. If $c_{ij} = \pm 1$, the correlation is said to be *perfect*. Each variable, of course, is perfectly correlated with itself ($c_{\alpha\alpha} = 1$ if $i = j = \alpha$).

The square root of a variance is called a *standard deviation* or root-mean-square (rms) amplitude; it is denoted by a prime (for example, $u_i' = (\overline{u_i^2})^{1/2}$). A characteristic velocity, or "velocity scale," of turbulence at some downstream position in a boundary layer might be defined as the mean rms velocity taken across the boundary layer at that position; in this way velocity scales used in dimensional analysis could be given a precise definition whenever desired.

Equations for the mean flow If we apply the decomposition rule (2.1.6) to the continuity equation (2.1.2), we obtain

$$\frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial}{\partial x_i}(U_i + u_i) = \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0. \tag{2.1.18}$$

If the average of all terms in this equation is taken, the last term vanishes because of (2.1.8, 2.1.10). Hence, the mean flow is incompressible:

$$\partial U_i / \partial x_i = 0. \tag{2.1.19}$$

Subtracting (2.1.19) from (2.1.18), we find that the turbulent velocity fluctuations are also incompressible:

$$\partial u_i / \partial x_i = 0. \tag{2.1.20}$$

The equations of motion for the mean flow U_i are obtained by substituting (2.1.6) and (2.1.12) into (2.1.1) and taking the average of all terms in the resulting equation. This yields, if all rules on averaging are observed (in particular, recall that $\partial U_i / \partial t = 0$),

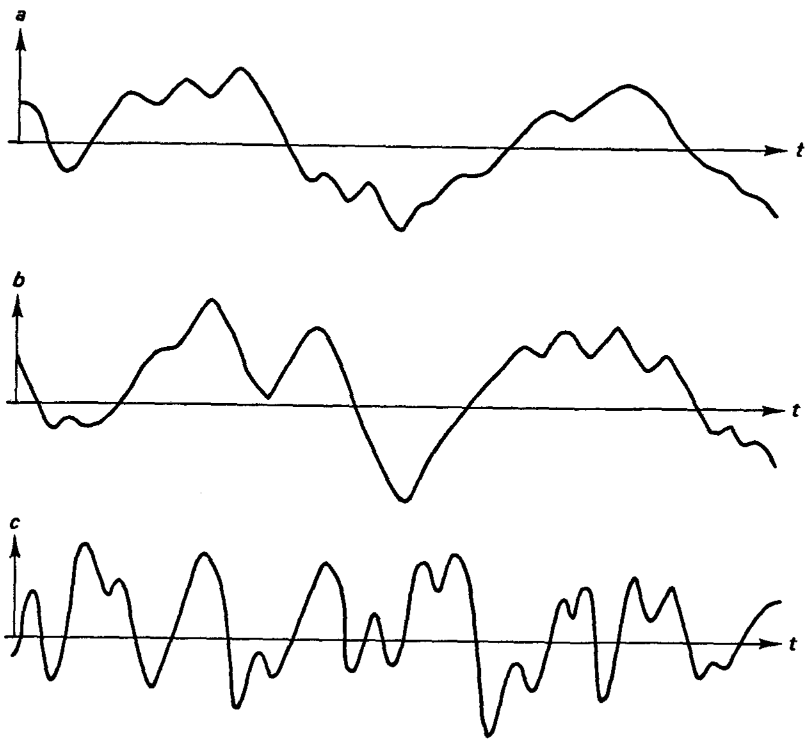

Figure 2.1. Correlated and uncorrelated fluctuations. The fluctuating variable a has the same sign as the variable b for most of the time; this makes $\overline{ab} > 0$. The variable c, on the other hand, is uncorrelated with a and b, so that $\overline{ac} = 0$ and $\overline{bc} = 0$ (note that $\overline{ab} \neq 0$, $\overline{ac} \neq 0$ does not necessarily imply that $\overline{bc} \neq 0$).

$$U_j \frac{\partial U_i}{\partial x_j} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \Sigma_{ij}. \tag{2.1.21}$$

With use of the continuity equation (2.1.20) for the turbulent velocity fluctuations, we may write

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial}{\partial x_j} \overline{u_i u_j}. \tag{2.1.22}$$

This term is analogous to the convection term $U_j\, \partial U_i/\partial x_j$; it represents the mean transport of fluctuating momentum by turbulent velocity fluctuations. If u_i and u_j were uncorrelated, there would be no turbulent momentum transfer. Experience shows that momentum transfer is a key feature of turbulent motion; the term (2.1.22) of (2.1.21) is not likely to be zero. Mean transport of fluctuating momentum may change the momentum of the mean flow, as (2.1.21) shows. The term (2.1.22) thus exchanges momentum between the turbulence and the mean flow, even though the mean momentum of the turbulent velocity fluctuations is zero ($\rho \overline{u_i} = 0$).

Because momentum flux is related to a force by Newton's second law, the turbulent transport term (2.1.22) may be thought of as the "divergence" of a stress. Because of the Reynolds decomposition, the turbulent motion can be perceived as an agency that produces stresses in the mean flow. For this reason, (2.1.21, 2.1.22) are rearranged, so that all stresses can be put together. This yields the *Reynolds momentum equation*:

$$U_j \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} (\Sigma_{ij} - \rho \overline{u_i u_j}). \tag{2.1.23}$$

If we recall that Σ_{ij} is given by (2.1.13), the total mean stress T_{ij} in a turbulent flow may be written as

$$T_{ij} = \Sigma_{ij} - \rho \overline{u_i u_j} = -P\, \delta_{ij} + 2\mu\, S_{ij} - \rho \overline{u_i u_j}. \tag{2.1.24}$$

The Reynolds stress The contribution of the turbulent motion to the mean stress tensor is designated by the symbol τ_{ij}:

$$\tau_{ij} \equiv - \rho \overline{u_i u_j}. \tag{2.1.25}$$

In honor of the original developer of this part of the theory, τ_{ij} is called the *Reynolds stress tensor*. The Reynolds stress is symmetric: $\tau_{ij} = \tau_{ji}$, as can be seen by inspection of (2.1.25). The diagonal components of τ_{ij} are normal stresses (pressures); their values are $\rho\overline{u_1^2}$, $\rho\overline{u_2^2}$, and $\rho\overline{u_3^2}$. In many flows, these normal stresses contribute little to the transport of mean momentum. The off-diagonal components of τ_{ij} are shear stresses; they play a dominant role in the theory of mean momentum transfer by turbulent motion.

The decomposition of the flow into a mean flow and turbulent velocity fluctuations has isolated the effects of fluctuations on the mean flow. However, the equations for the mean flow (2.1.23, 2.1.24) contain the nine components of τ_{ij} (of which only six are independent of each other) as unknowns additional to P and the three components of U_i. This illustrates the closure problem of turbulence. Indeed, if one obtains additional equations for τ_{ij} from the original Navier-Stokes equations, unknowns like $\overline{u_i u_j u_j}$ are generated by the nonlinear inertia terms. This problem is characteristic of all nonlinear stochastic systems.

This is a frustrating prospect. Therefore, many investigators have attempted to guess at a relation between τ_{ij} and S_{ij}. This is a tempting approach because the function of the Reynolds stress in the equations of motion seems to be similar to that of the viscous stress $2\mu S_{ij}$. We investigate the nature of possible relations between τ_{ij} and S_{ij} in Section 2.3; before this is done, some background material on the viscous stress is given in Section 2.2.

Turbulent transport of heat Turbulence transports passive contaminants such as heat, chemical species, and particles in much the same way as momentum. For later use, we develop the equation governing heat transfer in turbulent flow of a constant-density fluid. The density is approximately constant if temperature differences remain relatively small, if gravity-induced density stratification may be neglected, and if the Mach number of the flow is small.

The starting point is the diffusion equation for heat in a flow:

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\theta}}{\partial x_j} = \gamma \frac{\partial^2 \tilde{\theta}}{\partial x_j \partial x_j} \cdot \tag{2.1.26}$$

The thermal diffusivity γ is assumed to be constant; its dimensions are $\mathrm{m^2 sec^{-1}}$. The ratio ν/γ is called the Prandtl number.

The temperature $\tilde{\theta}$ at (x_i, t) is decomposed in a mean value Θ and temperature fluctuations θ, such that

$$\tilde{\theta} = \Theta + \theta, \tag{2.1.27}$$

$$\overline{\tilde{\theta}} \equiv \Theta = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \tilde{\theta}\, dt, \tag{2.1.28}$$

$$\overline{\theta} \equiv 0, \quad \partial\Theta/\partial t = 0. \tag{2.1.29}$$

The last condition has been imposed because time averages would not make sense in an unsteady situation.

Substituting (2.1.27) into (2.1.26) and taking the average of all terms in the resulting equation, we obtain

$$U_j \frac{\partial \Theta}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\overline{\theta u_j} + \gamma \frac{\partial \Theta}{\partial x_j}\right). \tag{2.1.30}$$

The mean heat flux Q_j per unit area and unit time in a turbulent flow then becomes (c_p is the specific heat at constant pressure)

$$Q_j = c_p\, \rho(\overline{\theta u_j} - \gamma\, \partial\Theta/\partial x_j). \tag{2.1.31}$$

The heat flux is thus a sum of the contributions of the molecular motion and of the turbulent motion. The analogy between (2.1.24) and (2.1.31) is striking; it is the analytical foundation for the belief that turbulence may transport heat in much the same way as momentum.

2.2 Elements of the kinetic theory of gases

In this section we discuss the molecular background of the viscosity and other molecular transport coefficients in dilute perfect gases (Jeans, 1940). For gases, the rudiments of kinetic theory are straightforward, but the kinetic theory of liquids is not nearly as well developed.

Pure shear flow Let us take a steady pure shear flow, homogeneous in the x_1, x_3 plane. The only nonvanishing velocity component is taken to be U_1; it is a function of x_2 only. If the flow is laminar, the only nonvanishing components of the viscous shear stress are

$$\sigma_{12} = \sigma_{21} = \mu\, \partial U_1/\partial x_2. \tag{2.2.1}$$

The flow situation corresponding to (2.2.1) is sketched in Figure 2.2.

The shear stress σ_{12} must result from molecular transport of momentum in the x_2 direction. Let v_1 and v_2 be the x_1 and x_2 components of the instantaneous velocity of a molecule relative to the mean flow. The x_1

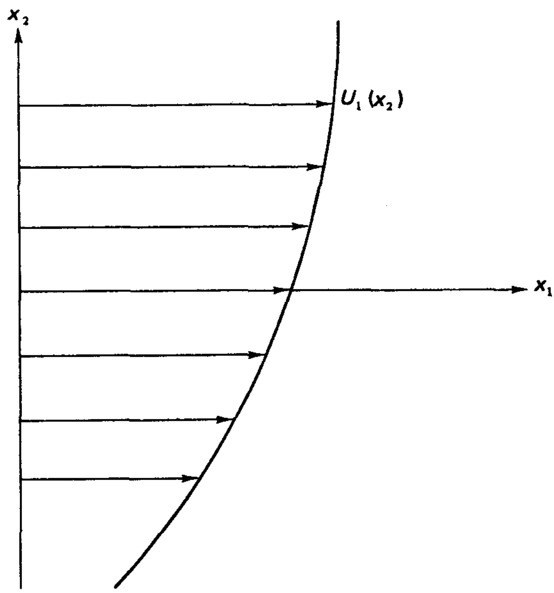

Figure 2.2. Pure shear flow. $U_2 = U_3 = 0$ and all derivatives with respect to x_1 and x_3 vanish.

momentum mv_1 of a molecule with mass m is transported in the x_2 direction if v_2 is correlated with v_1. The momentum transport per molecule is proportional to $m\, v_1 v_2$. If there are N molecules per unit volume, the transport of x_1 momentum in the x_2 direction is $Nm\, \overline{v_1 v_2}$ per unit time and area. Here, the overbar represents an average taken over a large number of molecules. Now, Nm is the mass per unit volume, which is the density ρ, and momentum flux per unit area and time may be equated with a stress. Hence,

$$\sigma_{12} = -\rho \overline{v_1 v_2}. \tag{2.2.2}$$

The minus sign in (2.2.2) is needed because positive values of v_2 should carry momentum deficit in a flow with positive σ_{12} and $\partial U_1/\partial x_2$. The analogy between (2.2.2) and the definition of the Reynolds stress given in (2.1.25) is intentional: a stress that is generated as a momentum flux can always be written as (2.2.2), no matter what mechanism causes the momentum flux.

Molecular collisions Kinetic theory of transport coefficients in gases estimates the right-hand side of (2.2.2) as follows. Suppose the mean free path (the average distance between collisions of molecules) is ξ. The unusual notation is selected because λ has to be reserved for one of the length scales occurring in turbulence. On the average, a molecule coming from $x_2 = -\xi$ collides with another molecule at the reference level ($x_2 = 0$). This process is illustrated in Figure 2.3. If we assume that because of this collision the molecule coming from below adjusts its momentum in the x_1 direction to that of its new environment, it has to absorb an amount of momentum equal to

$$M = m[U_1(0) - U_1(-\xi)]. \tag{2.2.3}$$

The quantity M is equal to the amount of momentum lost by the environment at $x_2 = 0$, because the upward-traveling molecule carries a momentum deficit with respect to the mean momentum $x_2 = 0$.

The right-hand side of (2.2.3) may be expanded in a Taylor series. This yields

$$M = m\xi \frac{\partial U_1}{\partial x_2} + \tfrac{1}{2} m\xi^2 \frac{\partial^2 U_1}{\partial x_2^2} + \dots . \tag{2.2.4}$$

The second and higher terms in the expansion may be neglected if

$$\frac{\partial U_1}{\partial x_2} \gg \tfrac{1}{2} \xi \frac{\partial^2 U_1}{\partial x_2^2}. \tag{2.2.5}$$

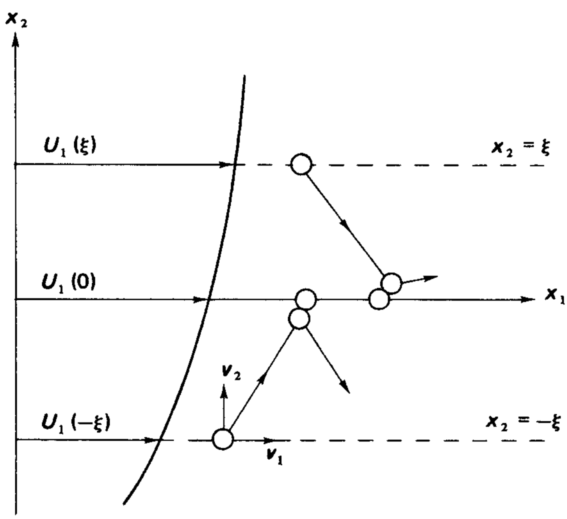

Figure 2.3. Molecular motion in a shear flow.

A local length scale ℓ of the flow $U_1(x_2)$ is defined as

$$\ell \equiv \frac{\partial U_1/\partial x_2}{\partial^2 U_1/\partial x_2^2}. \tag{2.2.6}$$

Hence, (2.2.5) may be written as

$$\ell >> \tfrac{1}{2}\,\xi. \tag{2.2.7}$$

For air at room temperature and density, $\xi = 7 \times 10^{-6}$ cm, so that for almost all flows the condition (2.2.7) is indeed satisfied. This implies that (2.2.4) may be approximated by

$$M = m\xi\, \partial U_1/\partial x_2. \tag{2.2.8}$$

In this simplified model, the quantity $\xi\, \partial U_1/\partial x_2$ is the part of v_1 that is correlated with v_2, apart from a minus sign needed due to the sign convention for σ_{12}. The number of collisions occurring at the reference level $x_2 = 0$ per unit area and time may be estimated as Na, where N again is the number of molecules per unit volume and a is the speed of sound (which is a good representative for the rms molecular velocity). If the momentum transfer per collision is M, the momentum transfer per unit area and time must be proportional to MNa. Using (2.2.8), we thus can write

$$\sigma_{12} = \alpha MNa = \alpha Nma\xi\, \partial U_1/\partial x_2. \tag{2.2.9}$$

Here, α is an unknown coefficient, which should be of order one. In air at ordinary temperatures and pressure, α is approximately $\frac{2}{3}$; we shall use this value for convenience.

Because $Nm = \rho$, (2.2.10) becomes

$$\sigma_{12} = \tfrac{2}{3}\rho a\xi\, \partial U_1/\partial x_2. \tag{2.2.10}$$

If we compare this with (2.2.1) and use $\mu = \rho\nu$, we obtain

$$\nu = \tfrac{2}{3}a\xi. \tag{2.2.11}$$

The Reynolds number formed with these variables is

$$\frac{a\xi}{\nu} = \frac{3}{2}. \tag{2.2.12}$$

That this Reynolds number turns out to be of the order one is no accident, because the viscosity is defined on the basis of molecular motion with velocity scale a and length scale ξ. The Reynolds number (2.2.12), however, is not a dynamically significant number because at length scales of order ξ the gas is not a continuum. For air at room temperature and pressure, $\xi = 7 \times 10^{-8}$ m, $a = 3.4 \times 10^2$ m/sec, so that $\nu = 15 \times 10^{-6}$ m^2/sec. It should be noted that elementary kinetic theory as given here cannot predict ratios of diffusivities (such as Prandtl number ν/γ).

Characteristic times and lengths The ratio of ξ to the local length scale ℓ of the flow is called the *Knudsen number K*. With (2.2.12), we obtain

$$K = \frac{\xi}{\ell} = \frac{3}{2}\frac{\nu}{a\ell} = \frac{3}{2}\frac{U}{a}\frac{\nu}{U\ell} = \frac{3}{2}\frac{M}{R}. \tag{2.2.13}$$

The Knudsen number is thus proportional to the ratio of the Mach number M and the Reynolds number R. In most flows $M \ll R$, so that the condition (2.2.7) is easily satisfied.

The Knudsen number is a ratio of length scales. The time scales involved in molecular transport of momentum are of interest, too. The molecular time scale is the time interval ξ/a between collisions; this is typically of the order of 10^{-10} sec. The time scale of the flow is the reciprocal of the velocity gradient $\partial U_1/\partial x_2$. If the velocity gradient is 10^4 sec^{-1}, corresponding to quite rapid shearing, the time scale of the flow is 10^{-4} sec. It is seen that changes in the flow are slow compared to the time scale representing molecular motion. This suggests that the thermal motion of the molecules should not be disturbed very much by the flow: molecules collide many thousands of times before the flow has advanced appreciably.

The correlation between v_1 and v_2 For future reference, it is useful to obtain some idea of how well the molecular velocity components v_1 and v_2 are correlated. The part of v_1 correlated with v_2 is proportional to $\xi\, \partial U_1/\partial x_2$, as shown by (2.2.8). Taking representative values for a rapid shearing flow in air ($\xi = 7 \times 10^{-8}$ m, $\partial U_1/\partial x_2 = 10^4$ sec^{-1}), we find that $\xi\, \partial U_1/\partial x_2 = 7 \times 10^{-4}$ m/sec. A correlation coefficient c between v_1 and v_2 may be defined as

$$c \equiv -\frac{\overline{v_1 v_2}}{(v_2')^2}. \tag{2.2.14}$$

Here, v_2' is the rms value of the x_2 component of the molecular velocity. As a comparison of (2.2.14) and (2.1.17) shows, we have used $v_1' = v_2'$. If we use the results previously given, we may estimate that

$$c \sim \frac{\xi \, \partial U_1/\partial x_2}{v_2'}. \tag{2.2.15}$$

Since v_2' is of the same order of magnitude as the speed of sound a, which is 3.4×10^2 m/sec for air at room conditions, we find that c is approximately 2×10^{-6}, indicating that v_1 and v_2 are very poorly correlated. If $\partial U_1/\partial x_2$ is estimated as U/ℓ, we find that the correlation coefficient is of the order of M^2/R, a parameter which indeed tends to be extremely small in most flows. We may conclude that the state of the gas is hardly disturbed by molecular momentum transfer. In other words, the dynamical equilibrium of the thermal motion of the molecules in shear flow of gases is, to a very close approximation, the same as the equilibrium state in a gas at rest. This implies that shear flow is not likely to upset the equation of state of the gas, unless M^2/R is large.

In anticipation of results that are obtained in Section 2.3, we note that the correlation coefficient of turbulent velocity fluctuations, defined in a manner similar to (2.2.14), is not small in turbulent shear flow. Consequently, the "state" of the turbulence is not independent of the mean flow field; on the contrary, the interaction between the mean flow and the turbulence tends to be quite strong.

Thermal diffusivity Molecular transport of scalar quantities is similar to the transport of momentum. The heat transfer rate is given by the second term of (2.1.31); in the model flow used here, the only nonvanishing component is

$$Q_2 = -\rho c_p \gamma \, \partial\Theta/\partial x_2. \tag{2.2.16}$$

In terms of molecular parameters, this is

$$Q_2 = -0.93 c_p \rho a \xi \, \partial\Theta/\partial x_2. \tag{2.2.17}$$

In this equation we have used (2.2.11) and $\nu/\gamma = 0.73$ (air at room conditions). The thermal diffusivity is larger than the diffusivity for momentum because molecules that travel faster than average carry more thermal energy with them and make more collisions per unit time. Energetic molecules thus do more than a proportional share in transporting heat.

2.3
Estimates of the Reynolds stress

We have seen that molecular transport can be interpreted fairly easily in terms of the parameters of molecular motion. It is very tempting to apply a similar heuristic treatment to turbulent transport. We again use a pure shear flow as a basis for our discussion. This flow is illustrated in Figure 2.4. Using (2.1.25) and (2.1.31), we find the rates of turbulent momentum transfer and heat transfer to be

$$\tau_{12} = -\rho\overline{u_1 u_2}, \tag{2.3.1}$$

$$H_2 = \rho c_p \overline{\theta u_2}. \tag{2.3.2}$$

The symbol H_2 is used to avoid confusion with the total rate of heat transfer Q_2.

Reynolds stress and vortex stretching Let us consider the Reynolds stress only. The existence of a Reynolds stress requires that the velocity fluctuations u_1 and u_2 be correlated. In a shear flow with $\partial U_1/\partial x_2 > 0$, negative values of u_1 should occur more frequently than positive ones when u_2 is positive, and vice versa. This is a rather intricate problem: the energy of the eddies has to be maintained by the shear flow, because they are continuously losing energy to smaller eddies. Molecules do not depend on the flow for their energy because the collisions between molecules are elastic. Eddies, on the

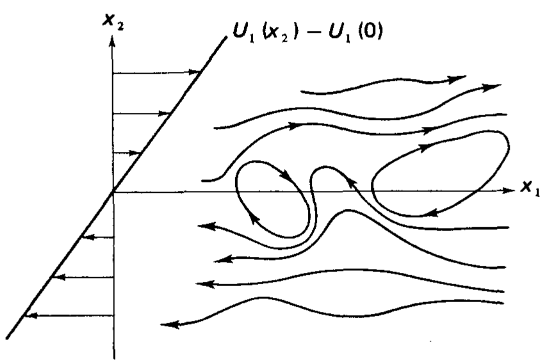

Figure 2.4. Turbulent pure shear flow. The mean velocity is steady: $U_2 = U_3 = 0$ and $U_1 = U_1(x_2)$. The instantaneous streamline pattern sketched refers to a coordinate system that moves with a velocity $U_1(0)$.

other hand, need shear to maintain their energy; the most powerful eddies thus are those that can absorb energy from the shear flow more effectively than others. Evidence (for example, Townsend, 1956, Bakewell and Lumley, 1967) suggests that the eddies that are more effective than most in maintaining the desired correlation between u_1 and u_2 and in extracting energy from the mean flow are vortices whose principal axis is roughly aligned with that of the mean strain rate. Such eddies are illustrated in Figure 2.5. The energy transfer mechanism for eddies of this kind is believed to be associated with vortex stretching: as the eddies in Figure 2.5 are being strained by the shear, conservation of angular momentum tends to maintain the good correlation between u_1 and u_2, thus allowing (as we discuss in more detail in Chapter 3) efficient energy transfer.

The interaction between eddies and the mean flow described here is essentially three dimensional. Two-dimensional eddies (velocity fluctuations without a component normal to the x_1, x_2 plane) may on occasion have appreciable Reynolds stress, but the mean shear tends to rotate and strain them in such a way that they would lose their capacity for extracting energy from the mean flow rather quickly.

These considerations suggest that a simple transport theory patterned after kinetic theory of gases is at best a very crude representation of reality. The dynamic interaction between the mean flow and the turbulence is too strong

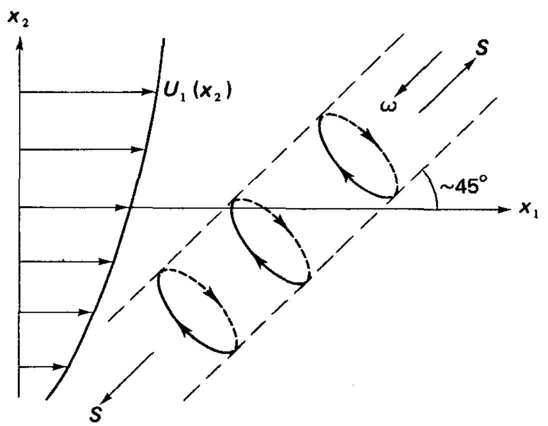

Figure 2.5. Three-dimensional eddies (vortices with vorticity ω) being stretched by the rate of strain S. The fluctuating velocity has strong components in the plane normal to the vorticity vector. Note that the shape of these eddies may differ widely from flow to flow.

to allow for a simple transport model. Also, a more detailed analysis of the energy and vorticity dynamics of the eddies (Chapter 3) is essential to the understanding of turbulence.

It should be noted that this discussion applies only to shear flows. If eddies receive energy in other ways (from buoyancy or a magnetic field, say), the picture may be entirely different.

The mixing-length model An estimate for the turbulent momentum flux can be obtained by analyzing the random motion of moving points ("fluid particles") in a turbulent shear flow. A formal treatment of the statistics of the motion of wandering points is given in Chapter 7; the less rigorous analysis presented here is more than adequate for a first look at turbulent transport.

Suppose a moving point starts from a level $x_2 = 0$ (see Figure 2.6) at time $t = 0$. Its x_1 momentum per unit volume is $\rho\tilde{u}_1(0, 0)$, where $\tilde{u}_1(0, 0)$ stands for the instantaneous velocity at $x_2 = 0$, $t = 0$. If we assume that the moving point does not lose its momentum as it travels upward, it has a momentum deficit $\Delta M = \rho\tilde{u}_1(x_2, t) - \rho\tilde{u}_1(0, 0)$ when it passes an arbitrary level x_2 at time t. Using the Reynolds decomposition of velocities, we can write the momentum deficit as

$$\Delta M = \rho[U_1(x_2) - U_1(0)] + \rho[u_1(x_2, t) - u_1(0, 0)]. \tag{2.3.3}$$

If the contribution of the turbulence to the momentum deficit can be neglected and if the difference $U_1(x_2) - U_1(0)$ may be approximated by

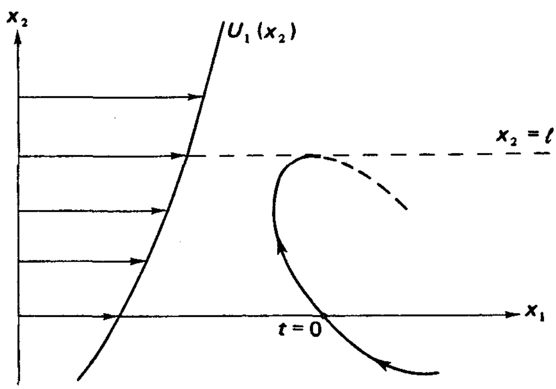

Figure 2.6. Transport of momentum by turbulent motion.

$x_2\ \partial U_1/\partial x_2$, where the gradient is taken at $x_2 = 0$, ΔM may be approximated by

$$\Delta M = \rho x_2\ \partial U_1/\partial x_2. \tag{2.3.4}$$

The volume transported per unit area and unit time in the x_2 direction is $\tilde{u}_2$ of the moving point. Now, $\tilde{u}_2 = dx_2/dt$, so that the average momentum flux at $x_2 = 0$ may be written as

$$\tau_{12} = \tfrac{1}{2}\rho \frac{\partial U_1}{\partial x_2} \frac{d}{dt}(\overline{x_2^2}). \tag{2.3.5}$$

The overbar here denotes an average over all moving points that start from $x_2 = 0$.

The dispersion rate $d\overline{(x_2^2)}/dt$ may be written as (see also Section 7.1)

$$\frac{d}{dt}(\overline{x_2^2}) = 2\,\overline{x_2 \frac{dx_2}{dt}} = 2\,\overline{x_2 u_2}. \tag{2.3.6}$$

If the fluid at any point did not continually exchange momentum with its environment, u_2 would remain constant for any given moving point, and $\overline{x_2 u_2}$ would continue to increase in time as x_2 increased. This is not realistic; instead, we expect that the correlation between u_2 and x_2 of a moving point decreases as the distance traveled increases. If we assume that u_2 and x_2 become essentially uncorrelated at values of x_2 comparable to some transverse length scale ℓ (see Figure 2.6), we may estimate that $\overline{x_2 u_2}$ is of order $u_2'\ell$. Here, u_2' is the rms velocity in the x_2 direction; the dispersion length scale ℓ is called the *mixing length*. Of course, this very estimate of $\overline{x_2 u_2}$ implies that momentum is not conserved when the moving point travels in the x_2 direction, so that this estimate makes the expression for the momentum deficit ΔM given in (2.3.4) very dubious, to say the least.

With $2\,\overline{x_2 u_2} = 2c_1 u_2'\ell$, (2.3.5) becomes

$$\tau_{12} = c_1 \rho u_2' \ell\, \partial U_1/\partial x_2. \tag{2.3.7}$$

The numerical coefficient c_1 is unknown.

We define the *eddy viscosity* ν_T (or turbulent exchange coefficient for momentum), in analogy with (2.2.1), by the equation

$$\tau_{12} \equiv \rho \nu_T\ \partial U_1/\partial x_2. \tag{2.3.8}$$

Comparing (2.3.7) and (2.3.8), we find that the eddy viscosity is given by

$$\nu_T = c_1 u_2' \ell. \tag{2.3.9}$$

If the mixing length ℓ and the velocity u_2' were known everywhere in the flow field and if the mixing-length model were accurate, the closure problem would be solved. The unknown Reynolds stress would be related to known variables and to the mean velocity gradient, making a solution of the equations of motion possible. However, the situation is not quite that simple. Even if we were willing to accept (2.3.7) as a model, u_2' and ℓ are not properties of the fluid but properties of the flow. This implies that u_2' and ℓ may vary throughout the flow field, making the eddy viscosity variable, dependent on the position in the flow. This is not a very promising prospect. Consequently, applications of (2.3.7) are usually restricted to flows for which it can be argued that u_2' is approximately constant (at least in the cross-stream direction) and for which ℓ is either constant or depends in a simple way on the geometry of the shear flow concerned.

In reality, turbulence consists of fluctuating motion in a broad spectrum of length scales. However, in view of the way ℓ occurs in (2.2.7), one may argue that large eddies contribute more to the momentum transfer than small eddies. The mixing-length model therefore favors large-scale motions; for simplicity, ℓ may be taken to be proportional to the size of the larger eddies.

The length-scale problem The approximations involved in the estimate (2.3.4) of the momentum deficit carried by a moving point need to be carefully considered. Because the distance over which momentum is transported is of order ℓ, the approximation (2.3.4) of (2.3.3) should be accurate over transverse distances of order ℓ. Let us define a local length scale $\mathscr{L}$ of the mean flow by (von Kármán, 1930)

$$\mathscr{L} \equiv \frac{\partial U_1/\partial x_2}{\partial^2 U_1/\partial x_2^2}. \tag{2.3.10}$$

The approximation $U_1(x_2) - U_1(0) = x_2\, \partial U_1/\partial x_2$ for all values of x_2 of order ℓ is valid only if

$$\mathscr{L} \gg \tfrac{1}{2}\ell. \tag{2.3.11}$$

In turbulent flows, however, the largest eddies tend to have sizes comparable to the width of the flow, as we have seen in Chapter 1. Consequently, ℓ is usually of the same order as the local length scale $\mathscr{L}$. This makes the "turbulent Knudsen number" $\ell/\mathscr{L}$ of order one. Note that both ℓ and $\mathscr{L}$ are transverse length scales: they are associated with the x_2 direction, which is normal to the mean flow.

We have to conclude that the truncation of the Taylor series expansion involved in (2.3.4) is not justified. Therefore, a *gradient-transport model*, which links the stress to the rate of strain at the same point in time and space, cannot be used for turbulent flow. It should be emphasized again that turbulence is an irreducible part of the flow, not a mere property of the fluid. Turbulence interacts strongly with its environment; the "state" of turbulence depends strongly on the flow in which it finds itself.

A neglected transport term The approximation (2.3.4) to the momentum deficit ΔM given by (2.3.3) neglected the contribution $\rho[u_1(x_2, t) - u_1(0, 0)]$. Let us call this $\rho\, \Delta u_1$. The momentum flux associated with this term is $\rho\, \overline{u_2 \cdot \Delta u_1}$, where the overbar again denotes an average over many moving points. The velocity difference Δu_1 should be very small for transverse distances small compared to ℓ, but it could be appreciable for values of x_2 of order ℓ, so that there is no a priori reason why this term can be neglected. However, in view of all of the other dubious assumptions involved in the mixing-length model, it does not seem useful to pursue this issue.

The mixing length as an integral scale In the derivation of (2.3.7), we used

$$\frac{1}{2}\frac{d}{dt}\overline{x_2^2} = \overline{x_2 u_2} = c_1\, u_2'\ell. \tag{2.3.12}$$

It is worthwhile to investigate how ℓ could be defined. For this purpose, consider how the value of x_2 increases as the moving point travels away from the reference level $x_2 = 0$. We can write

$$x_2(t) = \int_0^t u_2(t')\, dt'. \tag{2.3.13}$$

This implies that (2.3.12) may be written as (Taylor, 1921; see Friedlander and Topper, 1962)

$$\frac{1}{2}\frac{d}{dt}\overline{x_2^2} = \int_0^t \overline{u_2(t)u_2(t')}\, dt'. \tag{2.3.14}$$

The velocity $u_2(t)$ can be taken inside the integral because it is independent of t'; the averaging process can be performed on the integrand because it is done over many moving points, not over time.

In a statistically steady situation like the flow considered in this chapter, the origin of time is irrelevant, so that the correlation between $u_2(t)$ and $u_2(t')$ should depend only on the time difference $t - t' = \tau$. Let us define a correlation coefficient $c(\tau)$ by

$$c(\tau) \equiv \frac{\overline{u_2(t)u_2(t-\tau)}}{\overline{u_2^2}}. \tag{2.3.15}$$

Substituting (2.3.15) into (2.3.14), we obtain

$$\frac{1}{2}\frac{d}{dt}\overline{x_2^2} = \overline{u_2^2}\int_0^t c(\tau)\,d\tau. \tag{2.3.16}$$

The correlation coefficient $c(\tau)$ decreases as the time interval τ increases; at large values of τ the velocities $u_2(t)$ and $u_2(t')$ are uncorrelated. A sketch of $c(\tau)$ is shown in Figure 2.7.

The area under the curve in Figure 2.7 is given by

$$\mathcal{T} = \int_0^\infty c(\tau)\,d\tau; \tag{2.3.17}$$

it is assumed that $c(\tau)$ decreases rapidly enough at large τ to make $\mathcal{T}$ finite. The time $\mathcal{T}$ is called the *Lagrangian integral scale*. The adjective "Lagrangian" is used to indicate that it relates to moving points ("fluid particles"). The adjective "Eulerian" is used whenever correlations between two fixed points in a fixed frame of reference are considered. A more detailed discussion is given in Chapter 7.

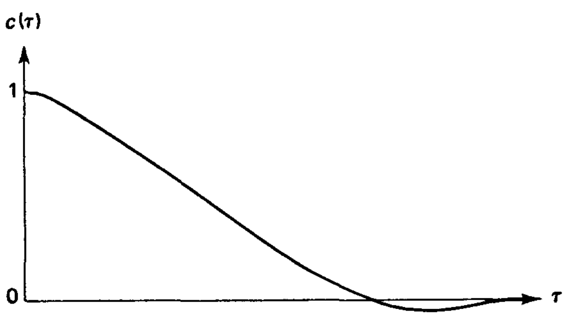

Figure 2.7. The Lagrangian correlation curve. Some correlation curves have negative tails, many do not.

Moving fluid loses its capability of transporting momentum when the correlation between x_2 and u_2 becomes zero. The time interval t involved in (2.3.16) should thus be large enough to make $c(t)$ zero. The dispersion rate then becomes (see also Section 7.1)

$$\frac{1}{2}\frac{d}{dt}\overline{x_2^2} = \overline{u_2^2}\,\mathcal{T}. \tag{2.3.18}$$

If we define (Taylor, 1921) a Lagrangian integral length scale ℓ_L by

$$\ell_L \equiv u_2'\mathcal{T}, \tag{2.3.19}$$

we can write (2.3.18) as

$$\frac{1}{2}\frac{d}{dt}(\overline{x_2^2}) = u_2'\ell_L. \tag{2.3.20}$$

The time scale $\mathcal{T}$ is hard to determine experimentally, because it requires that the motion of many tagged fluid particles be followed, say with photographic or radioactive tracer methods. In most turbulent flows, however, the length scale ℓ_L is believed to be comparable to the transverse Eulerian integral scale ℓ, which is defined by

$$\overline{u_2^2}\ell \equiv \int_0^\infty \overline{u_2(x_2)u_2(0)}\,dx_2. \tag{2.3.21}$$

The averaging process used in (2.3.21) is performed over a long period of time, with a fixed transverse separation x_2 and zero time delay between the two velocities. Experimental determination of ℓ is relatively simple.

If ℓ_L and ℓ are of the same order of magnitude, we thus may estimate $\overline{x_2u_2}$ as $c_1u_2'\ell$, where ℓ is defined by (2.3.21) (see also Sections 7.1 and 8.5).

The gradient-transport fallacy The mixing-length model has been discussed in great detail because of its ubiquitous use in much of turbulence theory. Let us now demonstrate that (2.3.7) is merely a dimensional necessity in a turbulent shear flow dominated by a single velocity scale u_2' and a single length scale ℓ.

The correlation coefficient c_{12} between u_1 and u_2 is defined as

$$c_{12} \equiv \overline{u_1u_2}/(u_1'u_2'). \tag{2.3.22}$$

Hence, we may write

$$\tau_{12} = -c_{12}\rho u_1'u_2'. \tag{2.3.23}$$

In all turbulent flows, u_1' and u_2' are of the same order of magnitude so that

(2.3.23) may be written as

$$\tau_{12} = c_2 \rho (u_2')^2. \tag{2.3.24}$$

In turbulent flows driven by shear, the unknown coefficients c_{12} and c_2 are always of order one: u_1 and u_2 are well correlated in eddies that can absorb energy from the mean flow by vortex stretching (Figure 2.5). Note, however, that in turbulence maintained in other ways, say by buoyancy, c_{12} and c_2 may be quite small.

The eddies involved in momentum transfer have characteristic vorticities of order u_2'/ℓ; they maintain their vorticity because of their interaction with the mean shear $\partial U_1/\partial x_2$. Let us write

$$u_2'/\ell = c_3 \, \partial U_1/\partial x_2, \tag{2.3.25}$$

so that c_3 is a nondimensional coefficient. If the straining of eddies is the effective mechanism that Figure 2.5 suggests it is, c_3 should be of order one. In effect, we are merely saying that the characteristic time of eddies (ℓ/u_2') and the characteristic time of the mean flow $(\partial U_1/\partial x_2)^{-1}$ should be of the same order if no other characteristic times or lengths are present, because turbulence is the fluctuating part of the flow. In particular, it is implied that ℓ and the differential length scale $\mathscr{L}$ defined in (2.3.10) are of the same order and that the mixing length is of the same order as the length scale of large eddies. The statement about time scales made here may be transposed into a statement about vorticities or strain rates if so desired: if $c_3 \sim 1$, (2.3.25) states that the vorticity found in the larger eddies is of the same order as the vorticity of the mean flow, and that the respective strain rates are also comparable.

If we use (2.3.25) to substitute for one of the u_2' occurring in (2.3.24), we find

$$\tau_{12} = c_2 c_3 \rho u_2' \ell \, \partial U_1/\partial x_2, \tag{2.3.26}$$

which, of course, is equivalent to (2.3.7). We see that we can relate the stress at $x_2 = 0$ to the mean velocity gradient at $x_2 = 0$ because the correlation between u_1 and u_2 is good and because the time-scale ratio is of order one. No conservation of momentum needs to be assumed; the mean-velocity gradient $\partial U_1/\partial x_2$ at $x_2 = 0$ may be used because it is a convenient representative of $\partial U_1/\partial x_2$ throughout an environment of scale ℓ. Indeed, (2.3.16) is only one member of a class of expressions

$$\tau_{12}(x_2 = 0) \sim \rho \, u_2' \ell \frac{\partial U_1}{\partial x_2} \qquad (\,|x_2| \leqslant \ell), \tag{2.3.27}$$

all of which are implied by (2.3.24) and (2.3.25). The localized estimate (2.3.26) merely is the most convenient member of this class. In other words, we may treat the local stress as if it were determined by the local rate of strain because there is only one characteristic length and one characteristic time. In short, (2.3.26) is a dimensional necessity that does not imply conservation of momentum or "localness" of the mechanism that produces the stress; (2.3.26) should not be mistaken for a gradient-transport postulate.

Further estimates Comparing (2.3.23) and (2.3.26), we see that the part of u_1 that is correlated with u_2 is of order $\ell\,\partial U_1/\partial x_2$. If the correlation between u_1 and u_2 is good and if u_1' and u_2' are of the same order, we may write

$$\tau_{12} = c_4 \rho \ell^2 \frac{\partial U_1}{\partial x_2} \left| \frac{\partial U_1}{\partial x_2} \right| . \tag{2.3.28}$$

In (2.3.28), c_4 is a coefficient of order one; the modulus of $\partial U_1/\partial x_2$ is used to make τ_{12} switch signs with $\partial U_1/\partial x_2$. This expression is the one originally proposed by Prandtl (see Hinze, 1959).

The eddy viscosity is of order $u_2'\ell$. The ratio of the Reynolds stress to the viscous stress is thus

$$\frac{\tau_{12}}{\mu\,\partial U_1/\partial x_2} = \frac{\nu_T}{\nu} = c_1 \frac{u_2'\ell}{\nu} = c_1 R_\ell . \tag{2.3.29}$$

This substantiates one of the results obtained in Chapter 1: the Reynolds number $u_2'\ell/\nu$ of the turbulent eddies may be interpreted as a ratio of diffusivities. In most flows, R_ℓ is very large, which implies that the Reynolds stress is much larger than the viscous stress. In other words, turbulent transport of momentum tends to be much more effective than molecular transport. If this is the case, the viscous terms in the equations for the mean flow may be neglected. The dependence of the mean flow on the Reynolds number is thus small, except in regions where ℓ and ν/u_2' are of the same order of magnitude.

Recapitulation We have found that, in a shear flow with one characteristic velocity and one characteristic length, the time scale of the turbulence is proportional to the time scale of the mean flow. Under certain circumstances, ℓ/u_2' may be as small as one-tenth of the reciprocal of $\partial U_1/\partial x_2$, but the general conclusion must be that turbulence in a shear flow cannot possibly be in a state of equilibrium which is independent of the flow field involved. The turbulence is continually trying to adjust to its environment, without ever succeeding. This conclusion is substantiated by the good correlation between u_1 and u_2. In all turbulent shear flows $|-\overline{u_1 u_2}| \sim 0.4 u_1' u_2'$; the value of 0.4 should be contrasted to the correlation coefficient for molecular motion, which was seen to be of order 10^{-6}. A theory for the Reynolds stress thus cannot be patterned after the kinetic theory of gases; the mixing-length model must be rejected, even though a mixing-length expression like (2.3.26) makes good dimensional sense in a situation where only one length scale and only one time scale are relevant.

In situations where more than one characteristic length and time are involved, the problem of the relation between stress and rate of strain generally becomes nearly intractable. If, for instance, the turbulence is mainly generated by buoyancy (as in an atmospheric boundary layer with an unstable temperature gradient), there is no need for the vorticity $\partial U_1/\partial x_2$ of the mean flow to be of order u_2'/ℓ, so that nothing can be said a priori about the value of the coefficient c_1 in (2.3.7). Problems such as this require a very careful study of the kinetic energy budget of turbulent motion.

In the model problem considered in this chapter, downstream variation in the flow was suppressed by virtue of the assumption that U_1 is only a function of x_2. In most flows, however, downstream changes do occur, introducing time scales such as the reciprocal of $\partial U_1/\partial x_1$ and length scales such as the distance x_1 from some suitably defined origin. These parameters would have to be taken into account were it not for the fact that in many flows of practical interest

$$\frac{\partial U_1}{\partial x_1} \ll \frac{\partial U_1}{\partial x_2}, \quad \ell \ll x_1. \tag{2.3.30}$$

If these inequalities hold almost everywhere in the flow, the downstream changes in the flow field are slow compared to the time scale of the turbulence, so that the turbulence may be in approximate equilibrium with respect to its environment at all values of the downstream distance x_1. This concept is vital to the theory of turbulent shear flows (Chapters 4 and 5).

2.4
Turbulent heat transfer

Passive contaminants are transported by turbulent motions in much the same way as momentum. The transfer of heat in the pure shear flow considered in this chapter is a good example. We assume here that the heat flux does not cause significant buoyancy effects.

Reynolds' analogy The vertical heat flux H_2 is given by (2.3.2):

$$H_2 = \rho c_p \overline{u_2 \theta}.$$

An eddy diffusivity for heat, γ_T, is defined by

$$H_2 \equiv -\rho c_p \gamma_T \, \partial\Theta/\partial x_2. \tag{2.4.1}$$

This is a mere definition, which does not assume anything about the nature of γ_T. In most turbulent flows, the "turbulent Prandtl number" ν_T/γ_T is close to one: turbulence transports heat just as rapidly as momentum (Hinze, 1959). Recall that τ_{12} may be expressed as (2.3.8):

$$\tau_{12} = \rho \nu_T \, \partial U_1/\partial x_2.$$

If ν_T/γ_T is equal to one, heat and momentum transfer are related by

$$\frac{H_2}{c_p \tau_{12}} = -\frac{\partial\Theta/\partial x_2}{\partial U_1/\partial x_2}. \tag{2.4.2}$$

This is called *Reynolds' analogy*. It is used to estimate the turbulent heat flux if the stress and the mean velocity and temperature fields are known. The analogy avoids an explicit statement on the magnitudes of the eddy diffusivities for heat and momentum, so that it can be applied even if ν_T and γ_T cannot be determined.

The mixing-length model Mixing-length theory (Taylor, 1915) estimates the heat flux as

$$H_2 = -\rho c_p c_5 u_2' \ell \, \partial\Theta/\partial x_2, \tag{2.4.3}$$

where c_5 is a coefficient of order one. The mixing-length model of turbulent heat transfer is not as misleading as the model of momentum transfer, because the temperature of a fluid particle is more nearly conserved than its momentum. Even so, (2.4.3), like its stress counterpart, does not need to be defended with a mixing-length model in order to justify its use in situations with a single characteristic length and velocity. If the correlation between u_2 and θ is good and if

$$\theta'/\ell \sim \partial\Theta/\partial x_2, \tag{2.4.4}$$

the heat transfer can be expressed as (2.4.3).

The assertion (2.4.4) may be understood as follows. Consider turbulent motion between $x_2 = 0$ and $x_2 = \mathscr{L}$, where $\mathscr{L}$ is the local length scale of the flow field, defined by (2.3.10). Let us assume that the mean temperature difference between $x_2 = 0$ and $x_2 = \mathscr{L}$ is $\Delta\Theta$. In turbulent flows, $\mathscr{L}$ and ℓ are of the same order of magnitude, so that the eddies, in attempting to mix the temperature field, create temperature fluctuations of order $\Delta\Theta$. This implies that $\theta' \sim \Delta\Theta$ if $\ell \sim \mathscr{L}$, which is expressed most concisely in the differential form (2.4.4). Strictly speaking, an average value of $\partial\Theta/\partial x_2$ between $x_2 = 0$ and $x_2 = \mathscr{L}$ should be used, but the definition of $\mathscr{L}$ implies that $\partial\Theta/\partial x_2$ is of the same order of magnitude everywhere between $x_2 = 0$ and $x_2 = \mathscr{L}$, so that a local value may be used to represent the average. It should be kept in mind, however, that a local interpretation of (2.4.3), though often convenient, is more restrictive than it needs to be.

The expression $\theta'/\ell \sim \partial\Theta/\partial x_2$ often is more reliable than its momentum counterpart $u_2'/\ell \sim \partial U_1/\partial x_2$, because the former merely expresses that turbulence mixes passive scalar contaminants over scales of order ℓ, whereas the latter is valid only if the turbulent motion is maintained by a mean strain rate. Momentum is not a passive contaminant; "mixing" of mean momentum relates to the dynamics of turbulence, not merely to its kinematics.

2.5
Turbulent shear flow near a rigid wall

Let us apply the concepts developed in this chapter to a pure shear flow in the vicinity of a rigid, but porous wall. The flow geometry is sketched in Figure 2.8. If there is no mass transfer (blowing or suction) through the wall, we shall find that there is only one velocity scale. In that case, mixing-length models may be used. However, if the mass-transfer velocity is different from zero, there are two velocity scales. We shall see that mixing-length theory cannot cope with that problem.

We take the mean flow to be steady and homogeneous in the x_1, x_3 plane. We take $U_3 = 0$ and $\partial P/\partial x_i = 0$ for $i = 1,2,3$. The flow may be thought of as occurring in a very wide channel, with the upper wall at $x_2 \to \infty$ moving at a certain velocity to maintain the momentum of the flow. The entire half-space $x_2 > 0$ is supposed to be filled with turbulent flow.

The equations of motion are

$$\frac{\partial U_2}{\partial x_2} = 0, \tag{2.5.1}$$

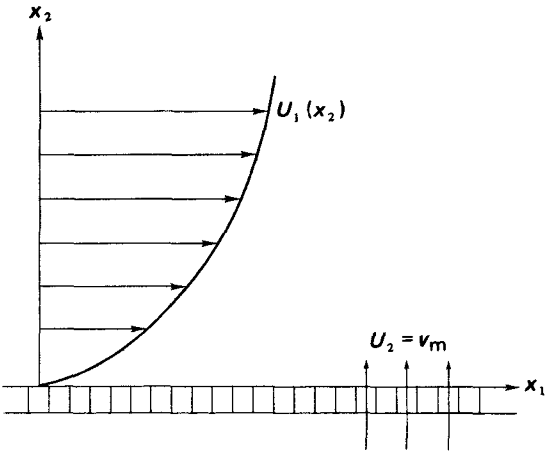

Figure 2.8. Turbulent flow near a rigid surface with mass transfer. The surface is at rest ($U_1(0) = 0$).

$$U_2 \frac{\partial U_1}{\partial x_2} = \frac{1}{\rho} \frac{\partial}{\partial x_2} T_{12}. \tag{2.5.2}$$

Equation (2.5.1) can be solved at once; because U_2 has to be independent of x_1 by virtue of downstream homogeneity, U_2 is uniform:

$$U_2 = v_m. \tag{2.5.3}$$

The mass-transfer velocity v_m is independent of x_1 and x_2, but it does not need to be zero as in the flow considered in Section 2.3.

With (2.5.3), (2.5.2) can be integrated to yield

$$\rho v_m U_1 = T_{12} - T_{12}(0). \tag{2.5.4}$$

The boundary condition $U_1(0) = 0$ is implied in (2.5.4). Let us define a *friction velocity* u_* by

$$T_{12}(0) = \rho u_*^2. \tag{2.5.5}$$

If the analysis is restricted to values of x_2 where $x_2 U_1/\nu >> 1$, the viscous contribution to the total shear stress T_{12} should be negligible, so that we may write

$$v_m U_1 = -\overline{u_1 u_2} - u_*^2. \tag{2.5.6}$$

A flow with constant stress If $v_m = 0$, the Reynolds stress $-\overline{u_1 u_2}$ is equal to u_*^2 at all values of x_2 for which viscous effects are negligible. A flow of this kind is called a *constant-stress layer*; it also occurs close to the wall in most turbulent boundary layers (Chapter 5). Assuming that u_1 and u_2 are well correlated, we conclude that u_2' must be independent of x_2 and proportional to u_*. The scale relation (2.3.25) between the vorticity of the turbulence and the vorticity of the mean flow becomes

$$u_*/\ell = \alpha_1 \, \partial U_1/\partial x_2, \tag{2.5.7}$$

in which α_1 is a coefficient of order one.

The rigid wall constrains the turbulent motion in the sense that transport of momentum downward from some level x_2 is restricted to distances smaller than x_2 itself. If no length scales are *imposed* on this flow, the only dimensionally correct choice for ℓ is

$$\ell = \alpha_2 x_2. \tag{2.5.8}$$

A comprehensive study of the implications of (2.5.8) is deferred until Chapter 5. With (2.5.8), (2.5.7) becomes

$$\partial U_1/\partial x_2 = u_*/\kappa x_2, \tag{2.5.9}$$

which readily integrates to

$$\frac{U_1}{u_*} = \frac{1}{\kappa} \ln x_2 + \text{const.} \tag{2.5.10}$$

The coefficient κ is known as the constant of von Kármán (Kármán constant, for short). Experiments have shown that κ is approximately equal to 0.4 (Hinze, 1959).

The additive constant in (2.5.10) is presumably determined by the no-slip condition ($U_1 = 0$ at $x_2 = 0$). However, this condition cannot be enforced because (2.5.10) is not valid at values of x_2 which are so small that the Reynolds number $x_2 U_1/\nu$ is of order unity.

In this flow without mass transfer through the surface, mixing-length models can be used because there is only one length scale (x_2) and one velocity scale (u_*), so that no ambiguity can arise. Specifically, (2.3.7) becomes

$$-\overline{u_1 u_2} = \kappa u_* x_2 \, \partial U_1/\partial x_2. \tag{2.5.11}$$

Because $-\overline{u_1 u_2}$ is equal to u_*^2 if $v_m = 0$, (2.5.11) produces (2.5.10) upon integration. Prandtl's version of the mixing-length formula can be applied with equal success.

Nonzero mass transfer If $v_m \neq 0$, the problem has two characteristic velocities, u_* and v_m. The length scale, however, remains proportional to x_2. This problem cannot be solved without making further assumptions. The least restrictive assumption we can make is that $\partial U_1/\partial x_2$ should be proportional to w/x_2, where w is an undetermined velocity scale that depends on u_* and v_m. Let us write

$$\partial U_1/\partial x_2 = w/x_2. \tag{2.5.12}$$

The numerical coefficient needed in (2.5.12) has been absorbed in the unknown velocity scale w.

Integration of (2.5.12) yields

$$U_1/w = \ln x_2 + \text{const.} \tag{2.5.13}$$

This equation is not a solution to the equations of motion; it is merely a consequence of the differential similarity law (2.5.12). Because w is unknown, it has to be determined experimentally. In this flow, v_m and u_* are the only two velocity scales, so that we may write

$$w/u_* = f(v_m/u_*). \tag{2.5.14}$$

Experimental results on w/u_* are given in Figure 2.9. In the case of blowing ($v_m > 0$), the Reynolds stress is larger than u_*^2; this results in an increase of w/u_*. If $v_m \gg u_*$, the friction velocity becomes relatively unimportant, so that w should be proportional to v_m. In the case of suction ($v_m < 0$), the Reynolds stress is smaller than u_*^2, so that w/u_* decreases. If the suction rate is large, the Reynolds stress becomes so small that turbulence cannot be maintained; this causes reverse transition from turbulent to laminar flow. If $v_m < 0$, the situation is further complicated by the fact that the suction imports not only mean momentum toward the wall but also turbulent kinetic energy.

The mixing-length approach The preceding analysis was based on the assumption expressed by (2.5.12). If the resulting velocity profile (2.5.13) is substituted into the equation of motion (2.5.6), there results

$$-\overline{u_1 u_2} = u_*^2 + v_m w(\ln x_2 + c). \tag{2.5.15}$$

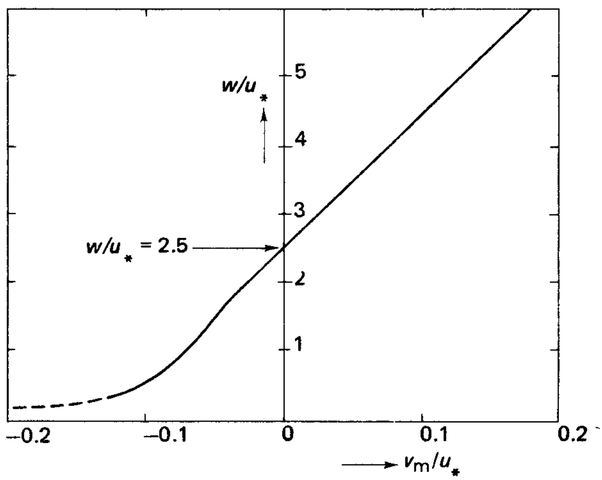

Figure 2.9. The velocity scale of flow near a rigid wall with mass transfer (based on data collected by Tennekes, 1965).

However, if we insist on using a mixing-length model and if we continue to use w as a characteristic velocity, we should write

$$-\overline{u_1 u_2} = \alpha_3 w x_2 \; \partial U_1 / \partial x_2 , \tag{2.5.16}$$

where α_3 is an unknown coefficient. If we substitute (2.5.12) into (2.5.16), we obtain

$$-\overline{u_1 u_2} = \alpha_3 w^2 . \tag{2.5.17}$$

A stress that is independent of x_2 is clearly not a correct solution: (2.5.6) states that the stress depends on x_2 because U_1 presumably depends on x_2. However, the difference between (2.5.15) and (2.5.17) is not as large as it seems. For $v_m = 0$, $w = 2.5\, u_*$ (Figure 2.9), so that $\alpha_3 = 0.16$. For small values of v_m/u_*, Figure 2.9 shows that $w/u_* = 2.5\ (1 + 9\ v_m/u_*)$, so that $\alpha_3 w^2$ may be approximated by $u_*^2 + 18\ v_m u_*$ if v_m/u_* is small. This is very much like (2.5.15) except for the suppressed dependence on x_2.

A third approach would be to substitute (2.5.16) into (2.5.6) without making a further substitution based on (2.5.12). Upon integration, this yields

$$-\overline{u_1 u_2} = v_m (\alpha_4 x_2)^{v_m/\alpha_3 w} . \tag{2.5.18}$$

This expression agrees neither with (2.5.15) nor with (2.5.17).

A fourth approach would be to use (2.5.12) to remove w from the mixing-length formula (2.5.16). This results in Prandtl's version of the mixing-length formula; after integration of the equation of motion (2.5.6) there results

$$-\overline{u_1 u_2} = [\alpha_5\, v_{\mathrm{m}} (\ln \alpha_6 x_2)]^2. \tag{2.5.19}$$

The corresponding velocity profile is obtained by substitution of (2.5.19) into (2.5.6). The proponents of (2.5.19) claim that it agrees with their experimental data. However, (2.5.19) contains two adjustable coefficients (α_5 and α_6), both of which may depend on v_{m}/u_*. Like (2.5.15), (2.5.17), and (2.5.18), (2.5.19) is not a solution to the equations of motion.

The limitations of mixing-length theory At this point it has become abundantly clear that mixing-length models are incapable of describing turbulent flows containing more than one characteristic velocity with any degree of consistency. None of the versions that were tried gives a clear picture of the roles of the two velocity scales; the effects of v_{m}/u_* on the integration constants remain altogether unresolved. Let us recall that mixing-length expressions can be understood as the combination of a statement about the stress $(-\overline{u_1 u_2} \sim w^2)$ and a statement about the mean-velocity gradient $(\partial U_1/\partial x_2 \sim w/x_2)$. These statements do not give rise to inconsistencies if there is only one characteristic velocity, but they cannot be used to obtain solutions to the equations of motion if there are two or more characteristic velocities that contribute to w in unknown ways. In other words, mixing-length theory is useless because it cannot predict anything substantial; it is often confusing because no two versions of it can be made to agree with each other. Mixing-length and eddy-viscosity models should be used only to generate analytical expressions for the Reynolds stress and the mean-velocity profile if those are desired for curve-fitting purposes in turbulent flows characterized by a single length scale and a single velocity scale. The use of mixing-length theory in turbulent flows whose scaling laws are not known beforehand should be avoided.

Problems

2.1 Consider a fully developed turbulent Couette flow in a channel between two infinitely long and wide parallel plane walls. The distance between the walls is $2h$, the lower wall is at rest and the upper wall moves with a velocity U_0 in its own plane. Assume that the flow consists of two wall layers (Section

2.5) which match at the center line of the channel. Find an expression for the friction coefficient at the lower wall ($c_f = 2u_*^2/U_c^2$, where U_c is the mean velocity at the center line) in terms of an appropriate Reynolds number. Estimate the additive constant in the logarithmic velocity profile (2.5.10) by assuming that near the walls there exist "viscous sublayers" in which the Reynolds number is so small that the Reynolds stress is negligible. The thickness of these sublayers is equal to $10\nu/u_*$. Sketch the velocity profile in the channel.

2.2 Experimental evidence obtained in pipe flow (Hinze, 1959) suggests that a more accurate representation of the velocity profile in turbulent Couette flow is obtained if it is assumed that the eddy viscosity is nowhere larger than $0.07hu_*$. Repeat the analysis of Problem 2.1 on this basis.

2.3 A certain amount of hot fluid is released in a turbulent flow with characteristic velocity u and characteristic length ℓ. The temperature of the patch is higher than the ambient temperature, but the density difference and the effects of buoyancy may be neglected. Estimate the rate of spreading of the patch of hot fluid and the rate at which the maximum temperature difference decreases. Assume that the size of the patch at the time of release is much smaller than ℓ and much larger than the Kolmogorov microscale η. The use of an eddy diffusivity is appropriate, but the choice of the velocity and length scales that are needed to form an eddy diffusivity requires careful thought, in particular as long as the size of the patch remains smaller than the length scale ℓ. In this context, a review of Problem 1.3 will be helpful.

2.4 A vortex generator in the shape of a low aspect-ratio wing is located on the wing of a Boeing 707. The height of the vortex generator is comparable to the thickness of the turbulent boundary layer over the wing. Give a qualitative description of the effect of the vortex generator on the momentum transfer in the boundary layer.