Lecture 3. Turbulent fluxes and TKE budgets (Garratt, Ch 2)

In this lecture…

- How does turbulence affect the ensemble-mean equations of fluid motion/transport?
- Force balance in a quasi-steady turbulent boundary layer.
- What are the sources and sinks of turbulent kinetic energy?

Hydrodynamic Equations of Turbulence

The ABL, though turbulent, is not homogeneous, and a critical role of turbulence is transport and mixing of air properties, especially in the vertical. This process is quantified using ensemble averaging (often called Reynolds averaging) of the hydrodynamic equations.

Boussinesq Equations (*G 2.2*)

For simplicity, we will use the Boussinesq approximation to the Navier-Stokes equations to describe boundary-layer flows. The ABL (like ocean BLs) obeys the two principal requirements for the accuracy of the Boussinesq approximation:

1. The ABL depth of O(1 km) is much less than the density scale height of O(10 km).
2. Typical fluid velocities are O(1-10 m s^{-1}), much less than the sound speed.

The Boussinesq equations of motion are:

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k}\times\mathbf{u} = -\frac{\nabla p'}{\rho_0} + b\mathbf{k}\left[+\nu\nabla^2\mathbf{u}\right], \text{ where buoyancy } b = g\theta_v'/\theta_0, \tag{3.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{3.2}$$

$$\frac{D\theta}{Dt} = S_\theta\left[+\kappa\nabla^2\theta\right], \quad S_\theta = -\frac{1}{\rho_0 c_p}\frac{\partial R_N}{\partial z} + L(C-E), \tag{3.3}$$

$$\frac{Dq}{Dt} = S_q\left[+\kappa_q\nabla^2 q\right], \quad S_q = E - C. \tag{3.4}$$

Here p' is a pressure perturbation, θ is potential temperature (defined using a reference pressure that is within 10% of the typical ABL air pressure), $q = q_v + q_l$ is mixing ratio (including water vapor q_v and liquid water q_l if present), and $\theta_v = \theta(1 + .608q_v - q_l)$ is virtual potential temperature including liquid water loading. S denotes a source/sink term, and ρ_0 and θ_0 are characteristic ABL density and potential temperature. κ and κ_q are the diffusivities of heat and water vapor. The terms involving molecular viscosity or diffusivity are in brackets because in most of the ABL, they are negligibly small compared to the other terms.

The most important source term for θ is divergence of the net radiative flux R_N (usually treated as horizontally uniform on the scale of the boundary layer, though this needn't be exactly true, especially when clouds are present). For boundary layers topped by stratocumulus or cumulus clouds, condensation C and evaporation E can be important sources of θ and q.

Ensemble Averaging (*G 2.3*)

Using mass continuity, the substantial derivative of any quantity a can be written in flux form:

$$Da/Dt = \partial a / \partial t + \nabla \cdot (\mathbf{u}a)\,. \tag{3.5}$$

This helps us write the ensemble average of Da/Dt as a sum of easily interpreted terms:

$$\begin{aligned}\overline{\frac{Da}{Dt}} &= \overline{\frac{\partial a}{\partial t}} + \nabla\cdot\overline{\mathbf{u}a} \\ &= \frac{\partial}{\partial t}\overline{(\bar a + a')} + \frac{\partial}{\partial x}\overline{(\bar u + u')(\bar a + a')} + \frac{\partial}{\partial y}\overline{(\bar v + v')(\bar a + a')} + \frac{\partial}{\partial z}\overline{(\bar w + w')(\bar a + a')} \\ &= \frac{\partial \bar a}{\partial t} + \nabla\cdot(\bar{\mathbf{u}}\bar a) + \frac{\partial}{\partial x}\overline{u'a'} + \frac{\partial}{\partial y}\overline{v'a'} + \frac{\partial}{\partial z}\overline{w'a'}\,. \end{aligned} \tag{3.6}$$

The three **eddy correlation** terms at the end of the equation express the net effect of the turbulence.

Consider a BL of characteristic depth H over a nearly horizontally homogeneous surface. The most energetic turbulent eddies in the boundary layer have horizontal and vertical lengthscale H and (by mass continuity) the same scale V for turbulent velocity perturbations in both the horizontal and vertical. The boundary layer structure, and hence the eddy correlations, will vary horizontally on characteristic scales $L_s >> H$ due to the impact on the BL of mesoscale and synoptic-scale variability in the free troposphere. If we let {} denote `the scale of', and assume $\{a'\} = A$, we see that *the vertical flux divergence is dominant*:

$$\left\{\frac{\partial}{\partial x}\overline{u'a'}\right\} = \frac{VA}{L_s} \ll \left\{\frac{\partial}{\partial z}\overline{w'a'}\right\} = \frac{VA}{H}\,. \tag{3.7}$$

Thus (noting also that $\nabla\cdot\bar{\mathbf{u}} = 0$ to undo the flux form of the advection of the mean),

$$\overline{\frac{Da}{Dt}} \approx \frac{\partial \bar a}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar a + \frac{\partial}{\partial z}\overline{w'a'} \quad \text{in the typical case } H << L_s. \tag{3.8}$$

If we apply (3.8) to the ensemble-averaged heat equation, and throw out horizontal derivatives of θ in the diffusion term using the same lengthscale argument $H << L_s$ as above, we find

$$\overline{\frac{D\theta}{Dt}} \approx \frac{\partial\bar\theta}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar\theta + \frac{\partial}{\partial z}\overline{w'\theta'} = \overline{S_\theta}\left[+\kappa\frac{d^2\bar\theta}{dz^2}\right], \text{ or}$$

$$\frac{\partial\bar\theta}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar\theta \approx -\frac{\partial}{\partial z}\overline{w'\theta'} + \overline{S_\theta}\left[+\kappa\frac{d^2\bar\theta}{dz^2}\right]. \tag{3.9}$$

Thus, the effect of turbulence on $\overline{\theta}$ is felt through the convergence of the vertical eddy correlation, or **turbulent flux** of θ. Except in the interfacial layer within mm of the surface, the molecular diffusion term in square brackets is negligible, so we will drop it from now except in situations it is clearly important. The **turbulent sensible and latent heat fluxes** are the turbulent fluxes of θ and q in energy units of W m^{-2}, $\rho_0 c_p \overline{w'\theta'}$ and $\rho_0 L \overline{w'q'}$ respectively. The surface turbulent fluxes (measured just above the interfacial layer) are given special symbols:

$$\text{Surface sensible heat flux (SHF)} \quad H_S = \rho_0 c_p \overline{w'\theta'}\Big|_{z=0^+} \tag{3.10}$$

$$\text{Surface latent heat flux (LHF)} \quad H_S = \rho_0 L \overline{w'q_v'}\Big|_{z=0^+} \tag{3.11}$$

If geostrophic wind (u_g, v_g)is defined in the standard way, the ensemble-averages of the horizontal components of the momentum equation (3.1) are:

$$\frac{\partial \overline{u}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{u} = f\left(\overline{v} - v_g\right) - \frac{\partial}{\partial z}\overline{u'w'} \, , \tag{3.12}$$

$$\frac{\partial \overline{v}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{v} = -f\left(\overline{u} - u_g\right) - \frac{\partial}{\partial z}\overline{v'w'} \, . \tag{3.13}$$

Often, the tendency and advection terms are much smaller than the two terms on the right hand side, and there is an approximate three-way force balance (see Figure 3.1) between turbulent momentum flux (often called **Reynolds stress**) convergence, Coriolis force and pressure gradient force in the ABL such that the mean wind has a component down the pressure gradient. The **cross-isobar flow angle** α is the angle between the near-surface wind and the geostrophic wind.

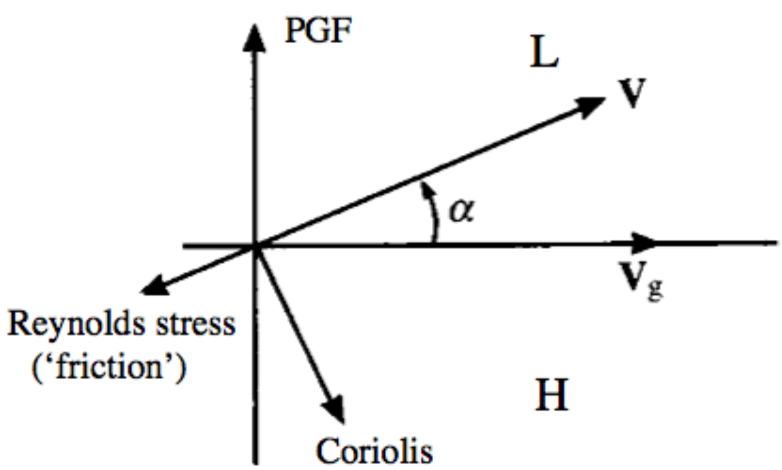

Fig. 3.1: Surface layer force balance in a steady state BL ($f > 0$). Above the surface layer, the force balance is similar but the Reynolds stress need not be along -**V**.

If the mean profiles of actual and geostrophic velocity can be accurately measured, the momentum flux convergence can be calculated as a residual in (3.12-13), and vertically integrated to deduce momentum flux. This technique was commonly applied early in this century, before fast-response, high data rate measurements of turbulent velocity components were perfected. It was not very accurate, because small measurement errors in either $\overline{\mathbf{u}}$ or $\mathbf{u}_g$ can lead to large relative errors in momentum flux.

In most BLs, the vertical fluxes of heat, moisture and momentum are primarily carried by large eddies with lengthscale comparable to the boundary layer depth, except near the surface where smaller eddies become important. This partitioning of turbulent fluxes across length or timescales can be quantified using the cospectrum. For instance, the **cospectrum** of w' and T', is the real part of the conjugate product of their Fourier transforms $\tilde{w}^*(\omega)\,\tilde{T}(\omega)$. If properly normalized, the cospectrum partitions the turbulent heat flux (the covariance of w' and T') across frequencies ω.

The cospectrum shown in Figure 3.2 is plotted in units of inverse period $n = \omega/2\pi$. It was calculated from tethered balloon measurements at two heights in the cloud-topped boundary layer we plotted in the previous lecture. It is positive, i. e. positive correlation between w' and T', at all frequencies, typical of a convective boundary layer. Most of the covariance between w' and T' is at the same inverse periods $n \sim 10^{-2}$ Hz that had the maximum spectral power in vertical velocity. Since the BL is blowing by the tethered balloon at the mean wind speed $\bar{U} = 7$ m s^{-1}, this frequency corresponds to large eddies of wavelength $\lambda = \bar{U}/n = 700$ m, comparable to the BL depth of 1 km.

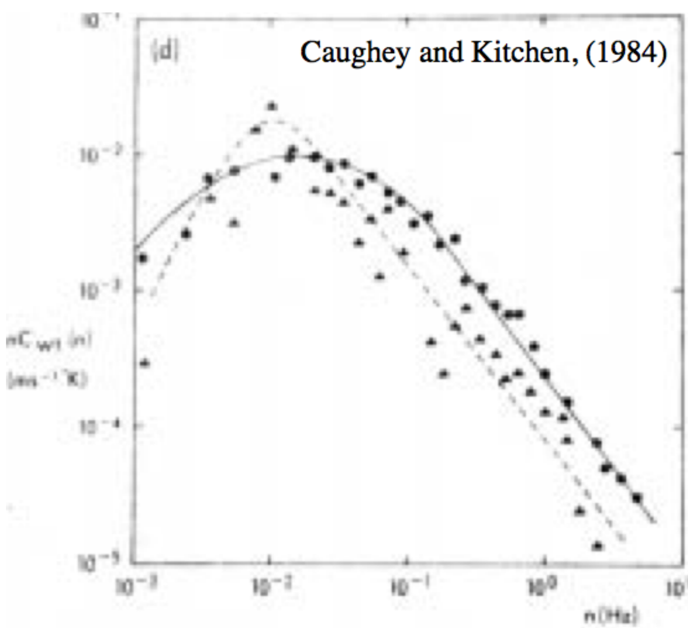

Fig. 3.3: Cospectrum of w' and T' at cloud base (triangles), top (circles) in convective BL.

Turbulent Kinetic Energy Equation (*G 2.5,6*)

To form an equation for ensemble-mean TKE $\overline{e} = \overline{\mathbf{u}' \cdot \mathbf{u}'} / 2$, we dot $\mathbf{u}'$ into the perturbation momentum equation and take the ensemble average. After considerable manipulation, we find that for the nearly horizontally homogeneous BL ($H << L_s$),

$$\frac{\partial \overline{e}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{e} = S + B + T + D , \qquad (3.14)$$

where

$$S = -\overline{u'w'}\frac{d\overline{u}}{dz} - \overline{v'w'}\frac{d\overline{v}}{dz} \qquad \text{(\textbf{shear production})}, \qquad (3.15)$$

$$B = \overline{w'b'} \qquad \text{(\textbf{buoyancy flux or production})}, \qquad (3.16)$$

$$T = -\frac{\partial}{\partial z}\left(\overline{w'e'} + \frac{1}{\rho_0}\overline{w'p'}\right) \qquad \text{(\textbf{transport} and pressure work)}, \qquad (3.17)$$

$$D = -\nu|\nabla \times \mathbf{u}|^2 \qquad \text{(\textbf{dissipation}, always negative, } -\varepsilon \text{ in Garratt)}. \qquad (3.18)$$

Shear production of TKE is the net conversion rate per unit mass from kinetic energy of the mean flow into TKE. Positive shear production occurs when the momentum flux is **downgradient**, i. e. has a component opposite ('down') the mean vertical shear. For this the eddies must tilt into the shear.

Buoyancy production of TKE is the net conversion rate per unit mass from gravitational potential energy of the mean state to TKE. Positive buoyancy production or flux occurs where relatively buoyant air is moving upward and less buoyant air is moving downward. It can be related to the sensible and latent heat fluxes as follows:

$$\overline{w'b'} = \frac{g}{\overline{\overline{T}}_\rho}\overline{w'T'_\rho} = g\left(\frac{1}{\overline{\overline{T}}}\overline{w'T'} + 0.61\overline{w'q'_v} - \overline{w'q'_l}\right) \qquad (3.19)$$

Assuming there is no liquid water flux at the surface, the surface buoyancy flux can thus be related to the surface sensible and latent heat fluxes

$$B_0 = \overline{w'b'}\Big|_{z=0^+} = \frac{g}{\rho c_p \overline{\overline{T}}}\underbrace{\left(SHF + \underbrace{0.61\frac{c_p\overline{T}}{L}}_{\approx 0.07}LHF\right)}_{\text{surface virtual heat flux } \overline{w'T'_v}(0^+)} \qquad (3.20)$$

Both S and B can be negative at some or all levels in the BL, but together they are the main source of TKE, so the vertical integral of $S + B$ over the BL is always positive.

The transport term T accounts for the redistribution of TKE due to the convergence of a mechanical energy flux that is the sum of the eddy flux of TKE and a pressure work term. Although this mainly moves TKE between different heights within the boundary layer, a small fraction of BL TKE can be lost to upward-propagating internal gravity waves excited by turbulence perturbing the BL top.

The dissipation term is the primary sink of TKE. It is proportional to enstrophy, which is dominated by the smallest (dissipation) scales in turbulent flows, so D can be considerable despite the smallness of ν.

Usually, the left hand side (the 'storage' term) is smaller than the dominant terms on the right hand side. Figure 3.3 shows typical profiles of these terms for a daytime convectively driven boundary layer and a nighttime shear-driven boundary layer. In the convective boundary layer, transport is considerable. Its main effect is to homogenizing TKE in the vertical. With vertically fairly uniform TKE, dissipation is also uniform, except near the ground where it is enhanced by the surface drag . Shear production is important only near the ground (and sometimes at the boundary layer top). In the shear-driven boundary layer, transport and buoyancy fluxes are small everywhere, and there is an approximate balance between shear production and dissipation.

The flux Richardson number

$$\mathrm{Ri}_f = - B/S \qquad (3.21)$$

characterizes whether the flow is stable ($\mathrm{Ri}_f > 0$), neutral ($\mathrm{Ri}_f \approx 0$), or unstable ($\mathrm{Ri}_f < 0$).

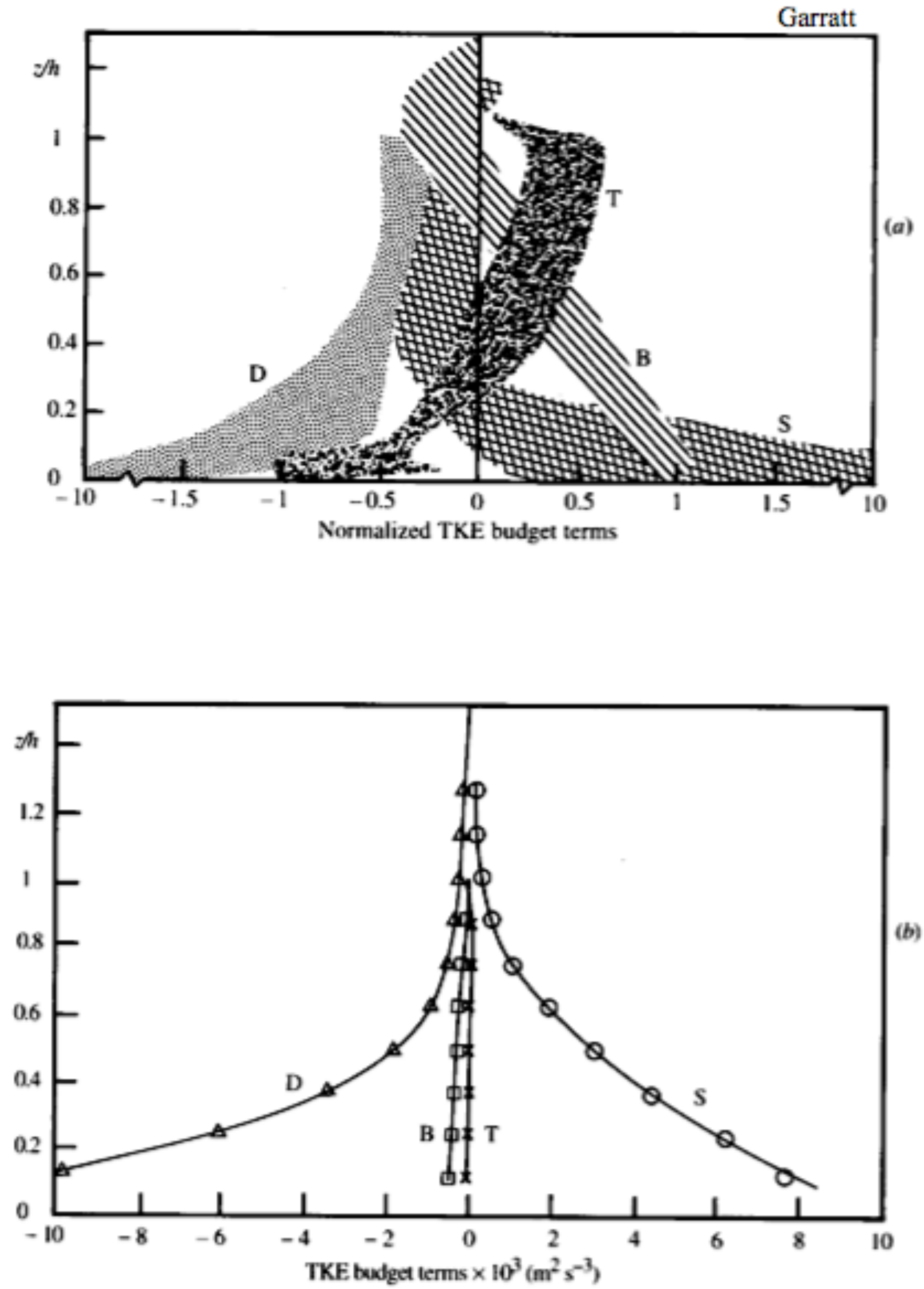

Fig. 3.3: Terms in the TKE equation (2.74b) as a function of height, normalized in the case of the clear daytime ABL (*a*) through division by w_*^3/h; actual terms are shown in (*b*) for the clear night-time ABL. Profiles in (*a*) are based on observations and model simulations as described in Stull (1988; Figure 5.4), and in (*b*) are from Lenschow *et al.* (1988) based on one aircraft flight. In both, B is the buoyancy term, D is dissipation, S is shear generation and T is the transport term. Reprinted by permission of Kluwer Academic Publishers.