OCN660 - Ocean Waves I

I. Introduction

Even with the long history of the study of waves, there is no satisfactory general definition of a "wave". A dictionary approach might be:

> Waves are any recognizable *disturbance* which advances through a *medium* with a *finite speed*.

Although many key words are used (disturbance, medium, finite speed), this definition is inadequate since it seems to exclude so-called ‘standing’ waves or ‘normal modes’, which are medium vibrations. Rather than focussing on a dictionary definition it’s more useful to list some of the main characteristics common to wave phenomena:

- waves transport *energy* from point-to-point in a medium;
- waves have *momentum* -- they exert a force when absorbed or reflected by an object;
- waves move with a *finite speed*, which is usually different from the speed of material particles comprising the medium;
- most *linear* water waves do not transport mass (a notable exception being oceanic Kelvin waves).

‘Small amplitude’ waves are waves in which the wave speed is much greater than the particle speeds. In the case of small amplitude waves, the governing (generally nonlinear) equations of motion can be replaced by *linear approximations*. An astonishing number of wave phenomena exhibit the mathematical property of *linearity* even though it is exactly true for only a few systems. The small amplitude assumption is frequently not a good one for ocean waves (e.g., surface gravity waves), but the linear approximation is a tremendous simplification mathematically. The mathematical theory of linear waves is relatively complete, whereas the nonlinear theory is in a more primitive state.

It is worth pointing out here that the great unification provided by the wave concept lies in the mathematics -- not in the physics. Even though two wave systems may have entirely different physics (e.g., ocean waves and charge oscillations of an electron) their mathematics may be quite similar.

This course has two objectives:

1. to survey some of the principal types of ocean waves; and,
2. to teach mathematical methods used to study all waves.

It is my hope that this course, combined with a suitable introductory course on fluid mechanics will provide the student with sufficient knowledge to readily access the detailed descriptions of oceanic wave types found in the books listed below.

From a balance of forces point of view, the principal oceanic wave types are:

- acoustic
- capillary
- gravity
- inertial or gyroscopic
- Rossby or vorticity

Each of these wave types exists in a hypothetical homogeneous ocean without horizontal boundaries or seafloor topography on a rotating spherical planet. These wave types can frequently combine to form hybrids, such as capillary-gravity waves. Many of these wave types have distinct additional forms upon the inclusion of more realistic oceanic features such as stratification, continental boundaries, and bottom topography. For instance, in a homogeneous ocean, gravity waves exist only through the gravitational restoring force acting on perturbations in sea surface height. But in a density stratified ocean, these surface gravity waves still exist with little modification due to the stratification, while a whole new set of waves (i.e., internal gravity waves) appears due to the gravitational restoring force acting on water parcels that have been displaced vertically from their equilibrium positions below the surface.

We will have time in this course only to recount the basic physics and kinematics of the last three of the five types listed above. But before we get to these waves, we must review (or introduce) some elementary mathematical & physical concepts. For additonal information on oceanic waves and the mathematical methods to study them, see the following books:

- General - Ocean Waves

LeBlond, P.H. and L.A. Mysak, 1978: *Waves in the Ocean*.

Gill, A.E., 1982: *Atmosphere-Ocean Dynamics*.

Kundu, P.K., 1990: *Fluid Mechanics*.

Lamb, H., 1932: *Hydrodynamics*.

• General - Wave Motion & Mathematical Methods

Johnson, R.S., 1997: *A Modern Introduction to the Mathematical Theory of Water Waves*.

Lighthill, J., 1978: *Waves in Fluids*.

Whitham, G.B., 1974: *Linear and Nonlinear Waves*.

• Short Period Surface Gravity Waves

Kinsman, B., 1965: *Wind Waves: Their generation and propagation on the ocean surface*.

Mei, C.C., 1992: *The Applied Dynamics of Ocean Surface Waves*.

• Long Period Surface Gravity Waves & Tides

Hendershott, M. C., 1981: Long waves and ocean tides, in *Evolution of Physical Oceanography*, B.A. Warren & C. Wunsch, editors.

Pugh, D.T., 1987: *Tides, Surges and Mean Sea Level*.

Parker, B.B. (ed.), 1991: *Tidal Hydrodynamics*.

• Atmospheric Gravity Waves

Nappo, C.J., 2002: *An Introduction to Atmospheric Gravity Waves*.

• Internal Gravity Waves

Munk, W., 1981: Internal waves and small-scale processes, in *Evolution of Physical Oceanography*, B.A. Warren & C. Wunsch, editors.

Phillips, O.M., 1977: *The Dynamics of the Upper Ocean*.

Turner, J.S., 1973: *Buoyancy Effects in Fluids*.

• Low-Frequency Waves - Rossby, Kelvin, Shelf, Equatorial

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*.

• Wave Interactions

Craik, A.D.D., 1985: *Wave Interactions and Fluid Flows*.

• Advanced Mathematics

Bender, C.M., and S.A. Orszag, 1978: *Advanced Mathematical Methods for Scientists and Engineers*

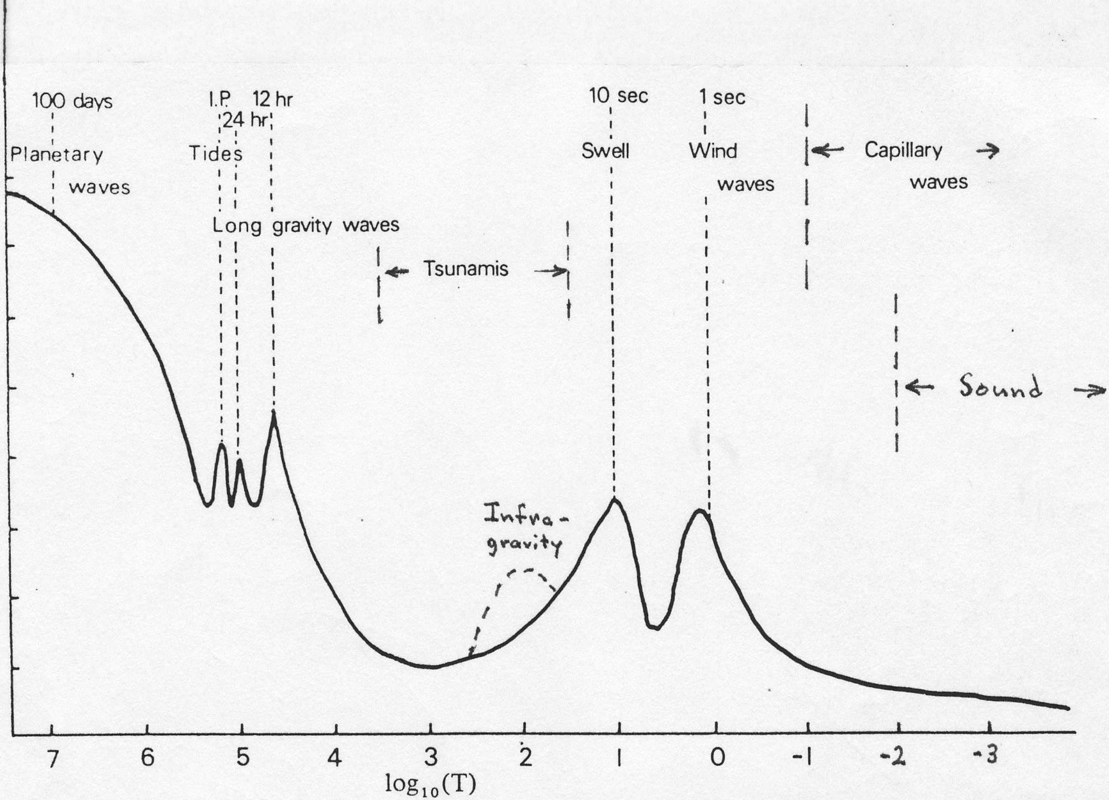


Fig. 2.1. Schematic energy spectrum of oceanic variability, showing the different types of waves occurring in the ocean. I.P. denotes the inertial period and is defined as $\pi/\Omega|\sin\phi|$, where Ω = magnitude of the Earth's rotation vector and ϕ is the geographic latitude (Section 3). In this picture I.P. = 35 hours, corresponding to a latitude of ± 20°. The relative amplitudes of the various parts of the spectrum do not necessarily reflect actual conditions. Taken from LeBlond & Mysak (1978).

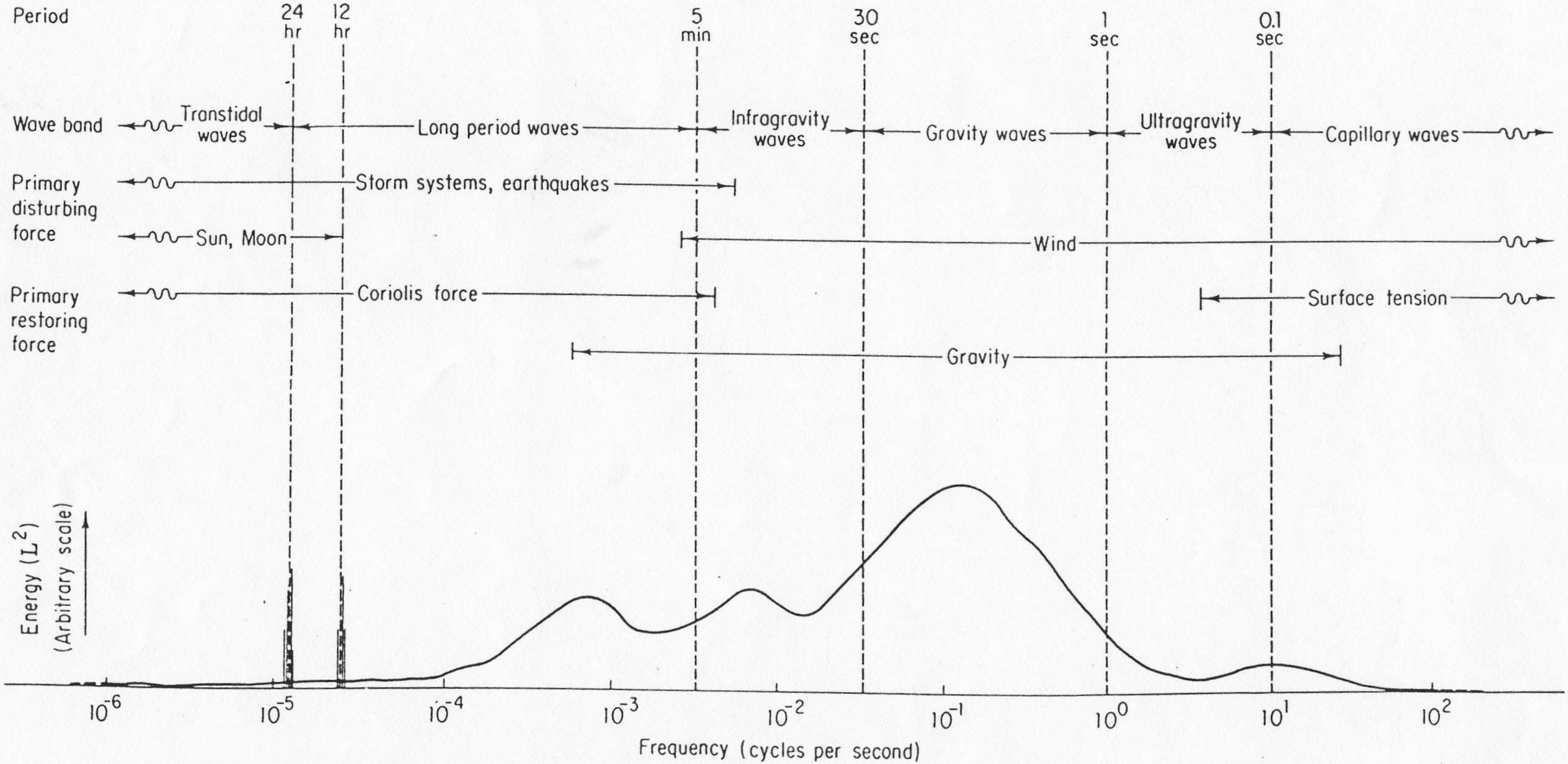


Fig. 1.2-1. Schematic (and fanciful) representation of the energy contained in the surface waves of the oceans—in fact, a guess at the power spectrum.

Kinsman, '65, Wind Waves

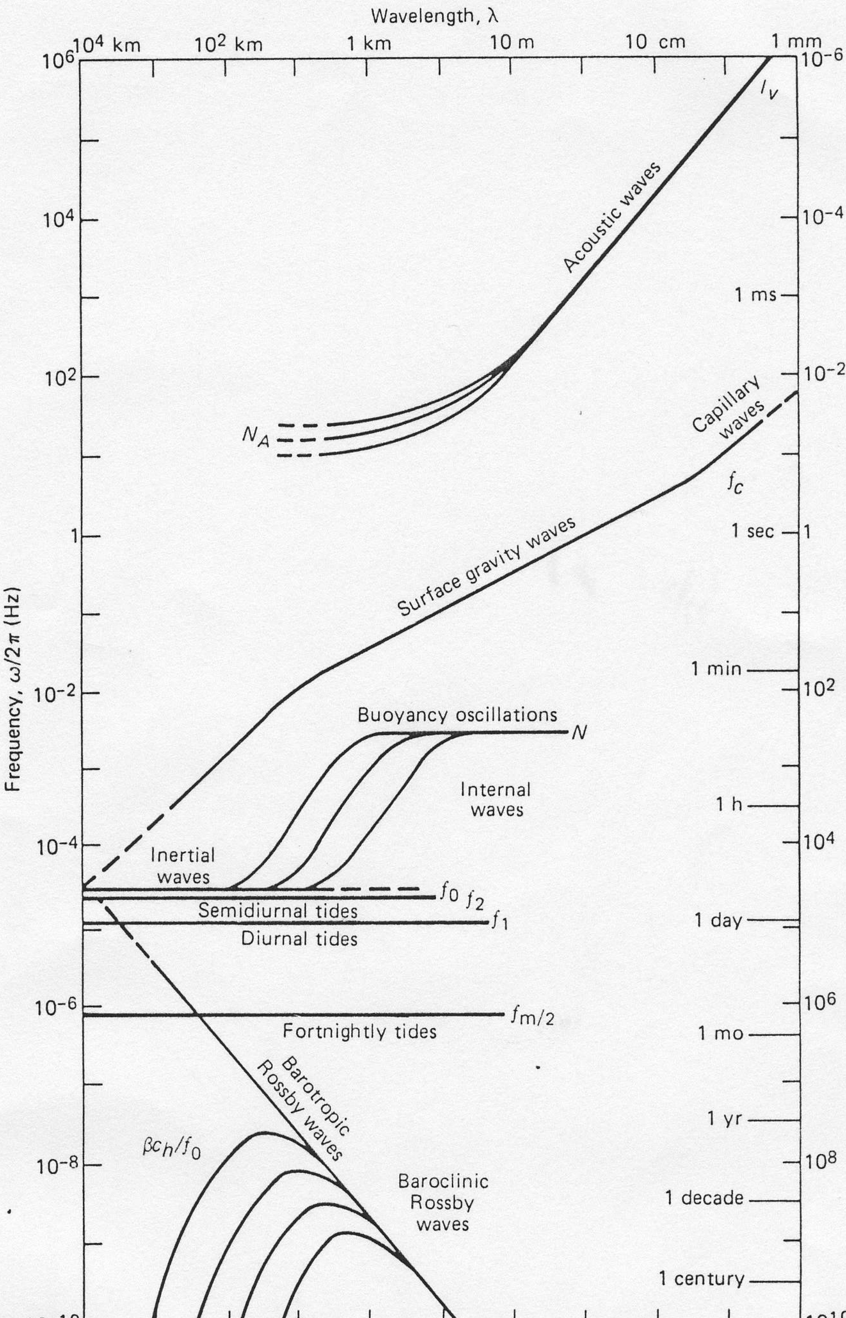


Fig. 6.54 Summary dispersion diagram for midlatitude oceanic waves, including Rossby, tidal, gravity, capillary, and acoustic waves in schematic form. The range of space and time shown includes most but not all of the relevant scales. Rossby wave numbers should be considered as negative. [Partly based on Hasselmann, K., *Prog. Oceanogr.* (1982).] Taken from Apel (1987)

II. The Simple Harmonic Oscillator (SHO)

[*Additional Reading*: Any good elementary text on classical mechanics, such as *Mechanics* by Symon.]

A. Introduction

The most important problem in one-dimensional motion, and fortunately one of the easiest to solve, is the harmonic or linear oscillator. The simplest example is that of a mass m fastened to a spring whose stiffness constant is k (Figure II.1). If we measure x from the relaxed position of the spring, then the spring exerts a restoring force (which we have already assumed is linear in x)

$$F = -kx\,. \tag{2.1}$$

The equation of motion for the mass m, assuming no other forces are acting, is given by Newton's Second Law, $F = m\ddot{x}$ (where the dots indicate two differentiations with respect to time), so that

$$\ddot{x} + \sigma^2 x = 0, \qquad \sigma^2 \equiv \frac{k}{m} \tag{2.2}$$

Eqn. (2.2) describes the free harmonic oscillator. "Free" means that the oscillator is not disturbed by external factors such as forcing, dissipation, coupling to other oscillators, etc. The solution to (2.2) can be written in complex notation as

$$x = Ae^{i\sigma t} + Be^{-i\sigma t}\,, \tag{2.3}$$

where $i^2 = -1$. A and B determine the *amplitude* of the oscillation and are found by requiring that the solution satisfies *initial conditions*. The solution in (2.3) is called a *free* mode of vibration with *natural frequency* σ. Note that a simple harmonic oscillator (SHO) such as described above does not display "wave" motion as previously defined. At best, it is analogous to a standing wave (or normal mode) in a two- or three-dimensional fluid (more on this later).

The complex notation in (2.3) can be avoided (remember that $\exp(i\sigma t) = \cos\sigma t + i \sin\sigma t$), but in general complex notation is much more convenient. Of course, in the real world one has to use real numbers. For real world problems the solution in (2.3) is actually incomplete and should read

$$x = \mathrm{Re}\left[Ae^{i\sigma t} + Be^{-i\sigma t}\right], \tag{2.4}$$

in which case $B = A^*$ leads to the simplest solution where the imaginary terms identically cancel (the asterisk denotes complex conjugate).

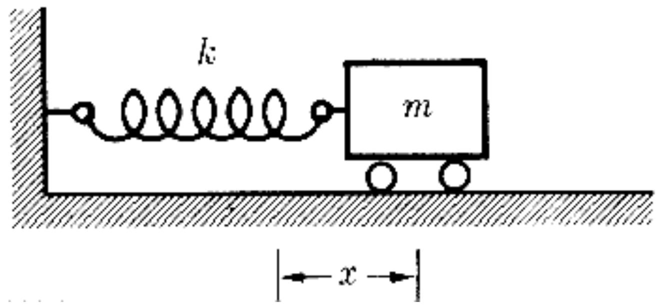


Figure II.1. Physical set up for a simple harmonic oscillator (SHO). A mass, *m*, rests on rollers and is attached to a wall by a spring with stiffness constant *k*. The motion of the oscillator is measured by the displacement, *x*, of the mass from its resting position.

The energy of the oscillator is a conserved quantity because there is no friction. We can prove this by multiplying (2.2) by $m\dot{x}$ and noticing that

$$m\,\dot{x}\,\ddot{x} + k\,\dot{x}\,x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left[m\,\dot{x}^2 + k\,x^2\right] = 0. \tag{2.5}$$

Thus $E \equiv (1/2)(m\,\dot{x}^2 + k\,x^2)$ is constant. E is the total energy and is the sum of the *kinetic* energy, $K \equiv m\,\dot{x}^2/2$, and *potential* energy, $V \equiv k\,x^2/2$.

B. Forcing & Dissipation

Now we add some more physics to our model by including both *forcing* and *dissipation*. The generalization of (2.2) is

$$\ddot{x} + \varepsilon\dot{x} + \sigma^2 x = f\ . \tag{2.6}$$

The physical significance of these new terms can be appreciated by forming the equation for the energy. One finds from (2.6) that

$$\dot{E} = -\varepsilon m\,\dot{x}^2 + m\,\dot{x}\,f\ . \tag{2.7}$$

The first term on the right hand side of (2.7) is dissipation - note how it systematically (that is, for all time) removes energy. For $f = 0$, the motion of the mass, at least for small damping, consists of a sinusoidal oscillation of gradually decreasing amplitude, as we'll see later. The frictional force in (2.6) is $-\varepsilon\dot{x}$ which is a resistance to motion proportional to velocity. At small speeds, particles moving through a fluid, such as air or water, experience a *drag* force proportional to their velocity and $\varepsilon\dot{x}$ might be a simple representation of this rubbing against the air or water. This is almost the only kind of frictional force for which the problem can be solved analytically. If the speed is large, the drag is proportional to the square of the velocity. In this case (2.6) is nonlinear:

$$\ddot{x} + \alpha\,|\dot{x}|\dot{x} + \sigma^2 x = f\ , \tag{2.8}$$

where $|\dot{x}|\dot{x}$ is used instead of $\dot{x}^2$ so that the force acts in the correct direction, i.e., against the motion. We'll explore later ways to extract useful information about the motion from such a nonlinear equation without actually solving for the motion.

The second term on the right hand side of (2.7) is forcing: $f(t)$ is a force that can excite the oscillator. $f(t)$ is some function of time that we specify, i.e., external forcing. If $f(t)$ is a sinusoidally varying force, (2.6) can lead to the phenomenon of *resonance*, where the amplitude of oscillation becomes very large, when the frequency of the impressed force equals the natural frequency of the undamped free oscillator.

Work is the application of a *force* over a distance. The *rate of working of a force* (or, *power*) is the product of that force times the velocity with which the point of application of the force moves. Both of the terms on the right hand side of (2.7) have this form.

B.1 Damped Oscillations

To illustrate the effects of damping, suppose that $f = 0$ in (2.6). Then we can solve this *homogeneous* equation by substituting $x = \exp(i\omega t)$. This gives a quadratic equation for the complex frequency ω:

$$\omega^2 - i\varepsilon\omega - \sigma^2 = 0 \quad , \tag{2.9a}$$

so

$$\omega_\pm = i\frac{\varepsilon}{2} \pm \sqrt{\sigma^2 - \frac{\varepsilon^2}{4}} \quad . \text{ For simplicity, let } \sigma_\varepsilon = \sqrt{\sigma^2 - \frac{\varepsilon^2}{4}} \ , \tag{2.9b}$$

and σ^2 is defined in (2.2). The quantity $\gamma \equiv \varepsilon/2$ is called the *damping coefficient*. For $\gamma < \sigma$, σ_ε is real and the oscillator is *underdamped*. With the two complex frequencies we can write the general solution as

$$x = \mathrm{Re}\left[B_+ e^{i\omega_+ t} + B_- e^{i\omega_- t}\right] = e^{-\gamma t}\ [P\cos(\sigma_\varepsilon t) + R\sin(\sigma_\varepsilon t)] \quad , \tag{2.10}$$

where the constants of integration B_+, P, etc. are determined by initial conditions. Figure II.2 illustrates how the oscillations in (2.10) decay under mild damping. In Figure II.2 the specific initial conditions have been chosen so that (2.10) reduces to $x = -A\, e^{-\gamma t} \cos(\sigma_\varepsilon t)$.

The solution (2.10) reduces to that in (2.4) when $\varepsilon \to 0$, but this limit is very subtle since even if $\varepsilon \ll 1$ the solutions in (2.4) and (2.10) differ significantly when $\varepsilon t \sim 1$, i.e., the friction is small but if it acts "long enough" it makes a significant difference.

To amplify this point consider an even more elementary example. The solution of

$$\dot{y} = -\varepsilon y \quad , \qquad y(0) = 1 \quad , \tag{2.11}$$

is $y(t,\varepsilon) = \exp(-\varepsilon t)$. Now $y(t,0) = 1$, but if $\varepsilon = 10^{-67} \ll 1$ it is not true that $y(t,10^{-67}) \cong y(t,0) = 1$, at least not if $t = 10^{67}$. With these simple examples we glimpse some of the difficulties confronting us if we neglect small terms in differential equations. The resulting approximations may be adequate for some length of time but then become inaccurate. This is known as a *secular* error.

For $\sigma = \gamma$, σ_ε in (2.9) is zero and (2.6) has only exponentially decaying solutions. For this situation the oscillator is said to be *critically damped*. For $\sigma < \gamma$, σ_ε is imaginary, yielding two solutions to (2.6) that decay at different rates. One decays faster than the case for $\sigma = \gamma$, while the other decays slower. These solutions are called *overdamped*. Figure II.3

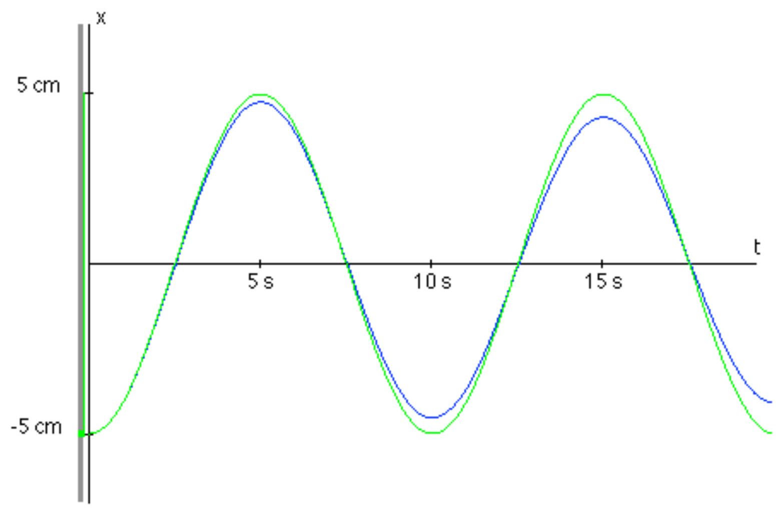


Figure II.2. The green curve shows the motion of the mass in Figure II.1, as a function of time, after the mass was displaced to x = -5 cm, held for a moment and released (in this case, the rollers are considered to be frictionless). The parameters, k and m, were chosen so that the natural frequency of the oscillator is approximately one cycle in 10 seconds. The blue curve indicates the motion of the oscillator under the assumption that there is a weak friction associated with the rollers that acts as a resistance force working in the direction opposite to the direction of motion.

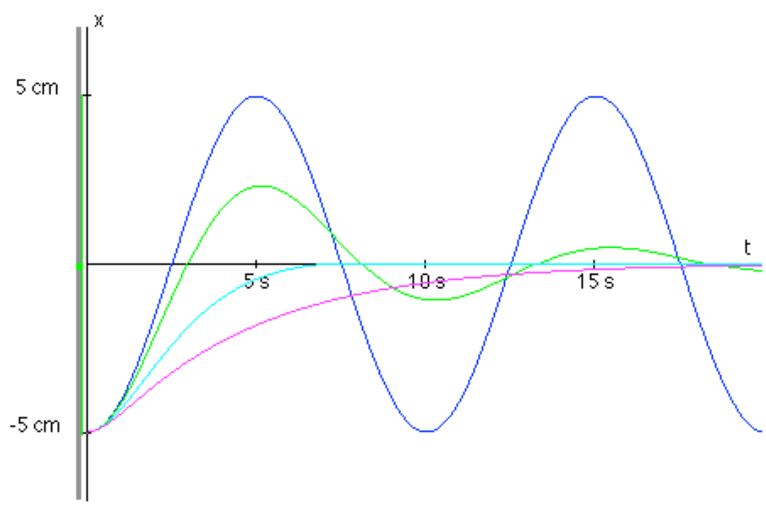


Figure II.3. Motion of the SHO in Figure II.1, but under the additional influence of different amounts of friction in the rollers acting as a resistance to the motion of the oscillator. The initial condition of the oscillator just before the motion began is described in Figure II.2. The blue curve shows the motion with no friction; the green curve shows moderate friction ($\gamma < \sigma$; the motion is *under-damped*); the cyan curve shows motion under *critical damping* ($\gamma = \sigma$); and, the magenta curve shows the motion when the oscillator is *over-damped* ($\gamma > \sigma$). Go to the following web site to experiment with an interactive applet simulating a damped harmonic oscillator: http://www.lon-capa.org/~mmp/applist/damped/d.htm.

schematically illustrates the difference between underdamped, critically damped, and overdamped motions. [An aside: For overdamped oscillations under continuing forcing in a fluid, the "wave" concept as defined in the Introduction is of minimal value, since the balance of forces is principally between the forcing and friction and since the "wave" loses most of its energy within one free period (as we'll see later). In this case, the time scale over which significant changes in position or velocity occur has little to do with the natural frequencies of the fluid, but rather is determined by the time scale over which the forcing varies.]

The non-dimensional ratio of σ divided by ε is commonly called the Q, or *resonance quality*, of the oscillator. Q was originally defined to be a measure of the ratio of time-averaged stored energy in the oscillator to the energy loss per cycle. It so happens that at critical damping Q is then 1/2. The Q is generally employed in the literature to communicate the relative importance of friction for the oscillations under consideration. A high Q implies weak dissipation, although what is considered "weak" is relative. For instance, a high Q in the ocean is 25, which can occur for surface and internal gravity waves, whereas to a seismologist that Q would be considered indicative of strong dissipation, since the seismic modes of the earth typically have Q's in the hundreds.

B.2 Forced Motion and Transients

As an example of the response of a SHO to forcing, let $\sigma = 1$, $\varepsilon = 0$ (no friction), and $f(t) = H(t)$, where

$$H(t) = 1 \text{ if } t > 0 \ , \qquad H(t) = 0 \text{ if } t < 0 \ , \tag{2.12}$$

is the Heaviside step function. This is a constant force that switches on suddenly at $t = 0$.

[N.b., there is no loss of generality in taking $\sigma = 1$ above, since the equation of motion (2.6) can be *non-dimensionalized* by defining new variables $\hat{t} \equiv \sigma t$ and $\hat{\varepsilon} \equiv \varepsilon/\sigma$.]

Under the assumption that the oscillator is at rest for $t < 0$, so that $x = \dot{x} = 0$ at those times, the solution to this problem is found by adding the particular solution of the inhomogeneous (2.6) to (2.6)'s homogeneous solutions, viz.,

$$x(t) = H(t)[1 - \cos(t)] \ , \qquad \dot{x} = H(t)\sin(t) \ . \tag{2.13}$$

With the simple forcing function in (2.12) there is a new equilibrium when $t > 0$: the differential equation $\ddot{x} + x = 1$ has a steady solution $x = 1$. In fact, the solution in (2.13) oscillates forever about this new equilibrium. In one of the homework problems you will show that if we now include friction this oscillation is damped so that $x \to 1$ as $t \to \infty$.

The energy of the forced oscillator is $E(t) = mx(t)$. Do you see why? Of course, you can verify this by direct substitution, but if you think about (2.7) you'll see there is a better way. Note that the energy is zero when $t = 0, 2\pi, \ldots$. Thus at these instants the external force has done no net work on the oscillator.

One other point to note about the solution in (2.13) is that both x and $\dot{x}$ are continuous at $t = 0$, but $\ddot{x}$ is discontinuous. This jump in the acceleration is required to balance the discontinuity in $f(t)$ on the right hand side of (2.6). Recall the general principle that higher derivatives of a function are increasingly singular. The "ramp" function $R(t) = tH(t)$ is continuous but its derivative $\dot{R}(t) = H(t)$ is discontinuous.

Now consider a slightly more complicated forcing function in (2.6) (we still persist with $\sigma = 1$ and $\varepsilon = 0$). Suppose that

$$f(t) = H(t) - H(t-T) \ , \tag{2.14}$$

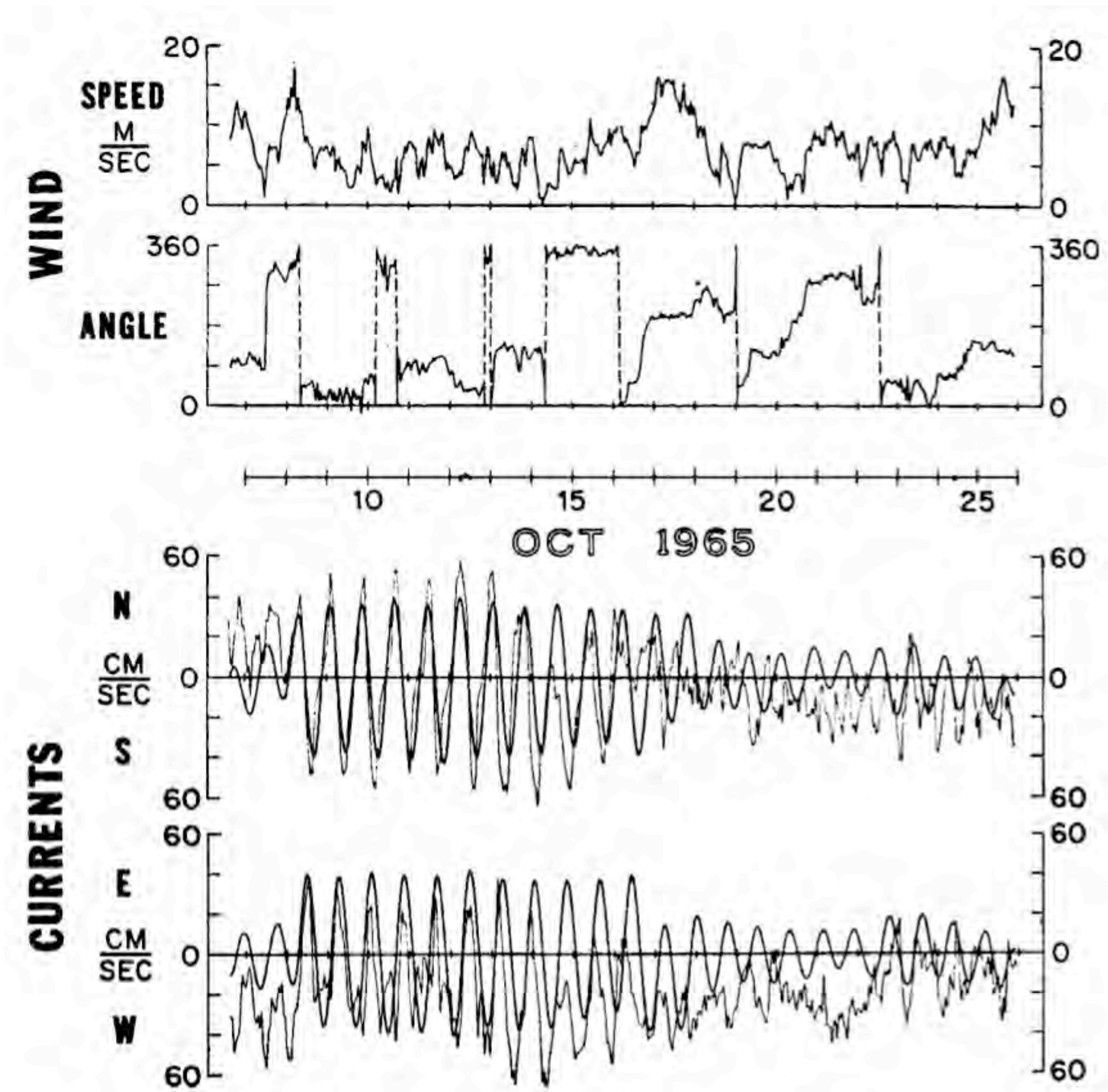
so that the force switches on at $t = 0$ and then switches off at $t = T$. (Draw a graph of $f(t)$ in (2.14) so you're clear on its structure.) One of the delightful characteristics of linear equations such as (2.6) is that we can use *linear superposition*. Thus the solution of (2.6) with the forcing in (2.14) is

$$x(t) = H(t)[1 - \cos(t)] - H(t-T)[1 - \cos(t-T)] \ , \tag{2.15}$$

i.e., a linear superposition based on the solution in (2.13). When $t > T$ both of the step functions in (2.15) are equal to 1 and the solution in (2.15) is

$$x(t) = \cos(t-T) - \cos(t) \ . \tag{2.16}$$

Depending on the size of T, we can arrange things so that $x(t) = 0$ (take $T = 2\pi, 4\pi, \ldots$), or $x(t) = -2\cos(t)$ (take $T = \pi, 3\pi, \ldots$). Because there is no friction the total work done by the force is equal to the energy left in the oscillator after the force stops acting. Thus in the first case the total work done by the force in (2.14) is zero, while in the second case the total work is $2m$. By varying T we can arrange things so that the total work done by the force is anywhere in between these two bounds: $0 \le E \le 2m$. The lesson from this example is that phase relationships matter: the amount of energy that an oscillator extracts from an external force depends not just on the size of the force, but also on how the force changes with time. The ocean provides some extraordinarily clear examples of this simple phenomenon. For example, surface winds force inertial-internal waves at the surface, and observations show how inertial waves started by one strong wind event can be readily halted by another wind pulse, leaving the ocean quiescent (see Pollard and Millard, *Deep-Sea Res*., 17, 813-821, 1970).



Extra Figure 1 - from Pollard & Millard (1970; *Deep-Sea Res*., **17**, 813-821). This figure provides a real world example of how an oscillator's motion can be stopped by appropriately timed forcing. This data was acquired from instruments moored in the ocean at 39^{o}20.5'N, 69^{o}59'W. At the top is a wind record (speed and direction). At the bottom are the north and east currents at 7 m, both measured (light line) and modeled (heavy line). The model is a simple damped harmonic oscillator intended to simulate just the dominant oscillations forced by the wind. The dominant oscillations are *inertial oscillations*, the dynamics of which are described later in Chap. 5. Note how two wind events (on Oct. 6 & 8) generate strong inertial oscillations that last for ~10 days with little decay and are then abruptly reduced in amplitude on Oct. 17 by an "unfortunately" timed wind event.

Extra Figure 2 - from Pollard & Millard (1970), with the same plotted variables as above, showing another example of how an appropriately timed wind forcing event (on Nov. 12) can halt the oscillation begun by an earlier wind event (on Nov. 10).

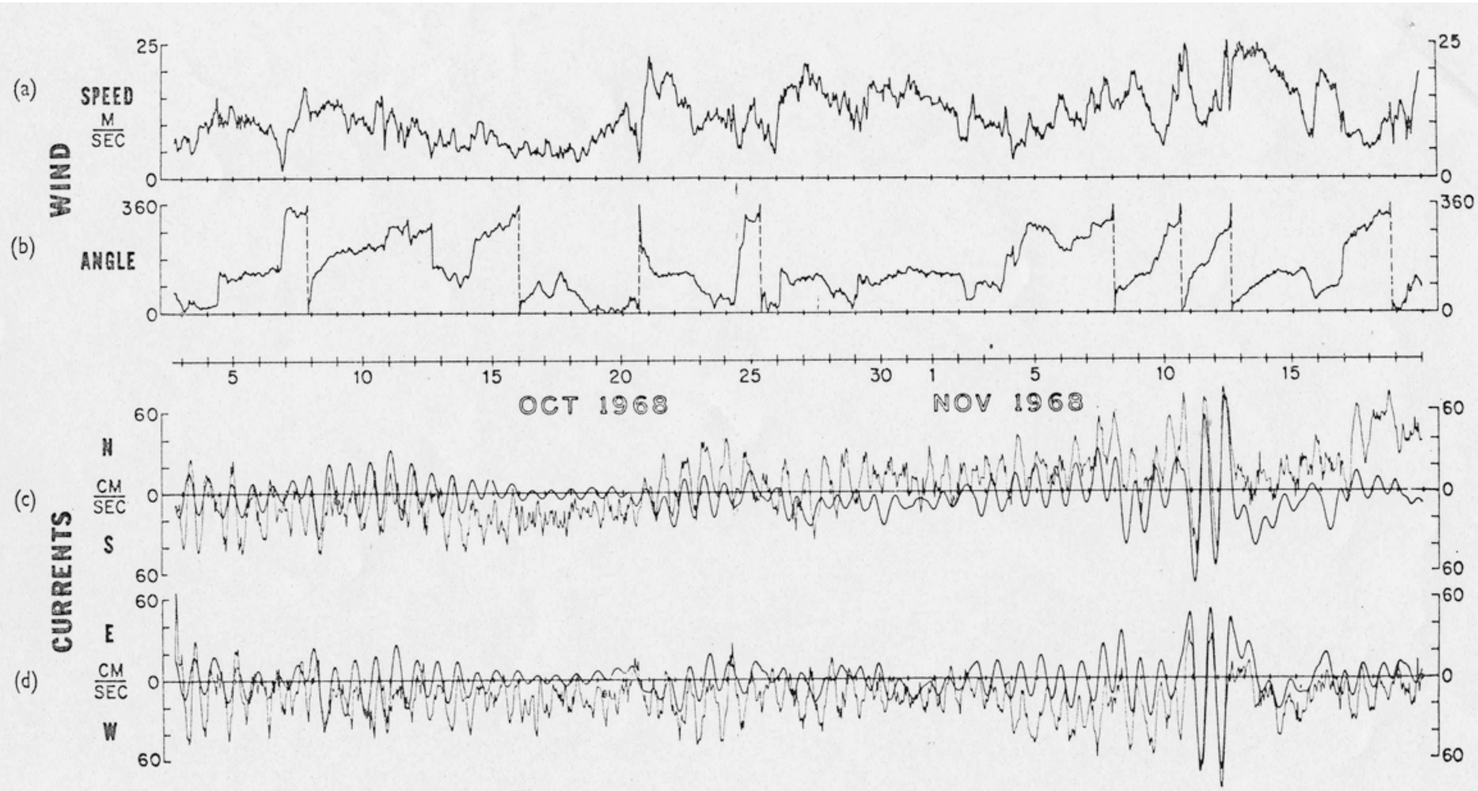


Fig. 2. Mooring 280. One-hour averages of (a) wind speed; (b) direction towards which wind blows (measured clockwise from north); (c) North–south components of the model with a 4-day damping (smooth, thick curve) and the observed current 2803 at 12 m deep (jagged, thin curve); (d) East–west components.

C. The Method of Averaging

In this subsection we use the energy equation in (2.7) to understand how "small" dissipation affects the free mode in (2.4). Of course we already know the answer to this question: the exact solution is given in (2.9) & (2.10): dissipation causes the mode to decay and the e-folding time of the amplitude is $2/\varepsilon$ (consequently, the e-folding time for the energy is $1/\varepsilon$). The process is illustrated in Figure II.2. Dissipation also changes the frequency of the oscillations. The frequency of the damped oscillations is σ_ε and when $\sigma \gg \varepsilon$ (the damping is weak) we have $\sigma_\varepsilon \cong \sigma - (\varepsilon^2/8\sigma)$. This slight shift in frequency is usually less important than the decay of the amplitude.

In this subsection we discuss how to calculate the effects of "small" dissipation. We will develop a simple technique for making approximate calculations and comparing these approximations with exact results such as (2.9)-(2.10). In the homework we discuss some more difficult problems where exact results either aren't available or are too complicated to be useful. The point of this is that it is possible to extract useful information from complicated equations without explicitly solving them.

What is meant by "small" dissipation? Note that there are two time scales in the problem: there is the period of the oscillator, which is essentially $2\pi/\sigma$, and the decay time $2/\varepsilon$. We suppose that these times are very different so that the oscillations are "fast" and their decay is "slow". Thus we suppose that

$$\sigma \gg \pi\,\varepsilon \tag{2.23}$$

so there are slowly decaying oscillations as shown in Figure II.2. Over one period of the oscillation the undamped solution in (2.4) is a good approximation of the weakly damped solution. But after many periods we have to adjust the amplitude to allow for the cumulative effects of small dissipation.

If we integrate the energy equation in (2.7) with $f=0$ over a period, from $t=0$ to $t=T\equiv 2\pi/\sigma$, we get

$$E(T) - E(0) = -\varepsilon m \int_0^T \dot{x}^2 \mathrm{d}t \ . \tag{2.24}$$

Of course we can use exactly the same procedure in the next period, $T<t<2T$, to calculate $E(2T)-E(T)$, and so on. In each period the decrease in energy will be proportional to the integrated kinetic energy in that period. Thus in the n'th period we can rewrite (2.24) as

$$\frac{E(nT) - E((n-1)T)}{T} = -\,\varepsilon m\,\overline{\dot{x}^2} \ , \tag{2.25}$$

where we have defined *the average over a period* of a function $f(t)$ as

$$\bar{f}(nT) \equiv \frac{1}{T}\int_{(n-1)T}^{nT} f(t_1)\,\mathrm{d}t_1 \ . \tag{2.26}$$

Since $E(t)$ changes only a little in the interval $0<t<T$, then $E \approx \bar{E}$ and the left hand side of (2.25) can be written

$$\frac{E(nT) - E((n-1)T)}{T} \approx \frac{\bar{E}(nT) - \bar{E}((n-1)T)}{T} \approx \frac{\mathrm{d}\bar{E}(nT)}{\mathrm{d}t} \ . \tag{2.27}$$

If $x = A\exp(i\sigma t) + A^*\exp(-i\sigma t)$, and A is almost constant in the interval $0<t<T$, then

$$\frac{\bar{E}}{m} = \frac{1}{2}\overline{\dot{x}^2} + \frac{1}{2}\sigma^2\overline{x^2} = \overline{\dot{x}^2} = \sigma^2\overline{x^2} = 2\sigma^2 AA^* \ . \tag{2.28}$$

[Fill in the steps in (2.28), noting that $\overline{\exp(i\sigma t)}=0$, and $\overline{[A\exp(i\sigma t) + A^*\exp(-i\sigma t)]^2} = 2AA^*$, etc. Other properties of the average are collected in the homework.] Hence at $t = nT$, (2.25) is

$$\frac{\mathrm{d}\bar{E}}{\mathrm{d}t} = -\varepsilon\bar{E} \ , \quad \text{or} \quad \bar{E} = E_0 e^{-\varepsilon t} \ , \tag{2.29}$$

so that the energy decays exponentially with an e-folding time of ε^{-1}. This means that the e-folding time of the amplitude is $2\varepsilon^{-1}$, in agreement with (2.10).

Notice that in (2.28) we showed by direct calculation that the time average of the kinetic energy is equal to the time average of the potential energy. This result is known as the *virial* or *equipartition theorem*. An alternative route to this important result is to multiply (2.6) by $x(t)$ and then time average:

$$\overline{x\ddot{x}} + \varepsilon\overline{x\dot{x}} + \sigma\overline{x^2} = \overline{xf} \ . \tag{2.30}$$

Using the results from the homework, the equation above simplifies to

$$-\overline{\dot{x}^2} + \sigma^2\overline{x^2} = \overline{fx} \ . \tag{2.31}$$

If $f = 0$ then the time average of the potential energy is equal to the time average of the kinetic energy.

The example above is a bit trivial - after all we can solve the problem exactly, so why bother with averaging? Consider a more difficult example, viz., weak quadratic damping as in (2.8),

$$\ddot{x} + \alpha|\dot{x}|\dot{x} + \sigma^2 x = 0 \ , \quad x(0) = 0 \ , \quad \dot{x}(0) = \delta \ . \tag{2.32}$$

Notice that the dimensions of α are (length)$^{-1}$ and the dimensions of δ are (length/time). First we non-dimensionalize (2.32)

$$\hat{t} = \sigma t \ , \quad \frac{\mathrm{d}}{\mathrm{d}t} = \sigma \frac{\mathrm{d}}{\mathrm{d}\hat{t}} \ , \quad \hat{x} = \frac{\sigma}{\delta} x \ , \quad \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} = \delta^{-1} \dot{x} \ . \tag{2.33}$$

In terms of non-dimensional variables the problem in (2.32) becomes

$$\frac{\mathrm{d}^2\hat{x}}{\mathrm{d}\hat{t}^2} + \varepsilon \left| \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} \right| \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}} + \hat{x} = 0 \ , \quad \hat{x}(0) = 0 \ , \quad \frac{\mathrm{d}\hat{x}}{\mathrm{d}\hat{t}}(0) = 1 \ , \tag{2.34}$$

where ε is non-dimensional and is given by

$$\varepsilon \equiv \frac{\alpha\delta}{\sigma} \ . \tag{2.35}$$

Henceforth, for simplicity the ˆ notation will be dropped.

The introduction of dimensionless variables in (2.33) is merely a change of notation, but it does tell us something very useful: the answer to the problem in (2.32) depends non-trivially only on one parameter, ε, and not three, α, σ and δ. Depending on the size of ε we may be able to approximately solve (2.34). In fact, we now consider the case when

$$\varepsilon \ll 1 \ , \tag{2.36}$$

and use the method of averaging to calculate how the oscillations slowly decay in this weakly damped limit.

The energy (per unit mass) equation for (2.34) is (returning to the ˙ notation for time derivative)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\varepsilon \, |\dot{x}| \dot{x}^2 \ , \quad E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \ . \tag{2.37}$$

The time average over one period of (2.37) is

$$\frac{\mathrm{d}\bar{E}}{\mathrm{d}t} = -\varepsilon \, \overline{|\dot{x}| \dot{x}^2} \ . \tag{2.38}$$

If $\varepsilon = 0$ the solution of (2.34) is

$$x = A \sin(t) \ , \quad E = \bar{E} = \frac{A^2}{2} \ , \tag{2.39}$$

where $A = 1$. If $0 < \varepsilon \ll 1$ then the amplitude A is a function of time, i.e, $x(t) = A(t)\sin(t)$ with the initial condition $A(0) = 1$. The time dependence of A is calculated from the energy balance in (2.38). To do this we have to express the right hand side of (2.38) in terms of $A(t)$:

$$\begin{aligned}
\overline{|\dot{x}|\,\dot{x}^2} &= \frac{1}{2\pi}\int_0^{2\pi} A^3(t)\,|\cos(t)|\,\cos^2(t)\,\mathrm{d}t\ , \\
&\approx A^3\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos^3(t)\,\mathrm{d}t\ , \\
&= \frac{4A^3}{3\pi}\ , \\
&= \frac{8\sqrt{2}\bar{E}^{3/2}}{3\pi}\ .
\end{aligned} \tag{2.40}$$

Notice that in the manipulation above we have treated $A(t)$ as a constant (i.e., it has been pulled outside the integral). This approximation is fine provided that the oscillations decay slowly so that there is very little change in $A(t)$ over one period.

Using (2.40), (2.38) is

$$\frac{\mathrm{d}\bar{E}}{\mathrm{d}t} = -c\ \varepsilon\bar{E}^{3/2}\ , \quad \text{or} \quad \bar{E} = E_0\,[1 + \frac{c\,\varepsilon t}{2}\,E_0^{1/2}]^{-2}\ , \tag{2.41}$$

where $c = 8\sqrt{2}/3\pi$. Thus, when $\varepsilon t \gg 1$ the amplitude of the oscillations ultimately decays like t^{-1}.

D. Forced Oscillations and Resonance

D.1 Sinusoidal Forcing Functions

In this section we discuss the solution of (2.6) with a particularly important forcing function; that is, we wish to solve

$$\ddot{x} + \varepsilon\dot{x} + \sigma^2 x = f\ , \tag{2.42a}$$

with the sinusoidal forcing

$$f = Fe^{i\omega_f t} + F^* e^{-i\omega_f t} = A\cos(\omega_f t) + B\sin(\omega_f t)\ \ . \tag{2.42b}$$

For $F = a + bi$, $A = 2a$ and $B = -2b$. To solve (2.42a) we look for a solution which is the sum of a homogeneous and a particular solution

$$x = x_{\mathrm{H}} + x_{\mathrm{P}}\ \ . \tag{2.43}$$

This works because (2.42a) is a *linear* system.

A particular solution can be found by substituting

$$x_{\mathrm{P}} = A_{\mathrm{P}}e^{i\omega_f t} + A_{\mathrm{P}}^* e^{-i\omega_f t} \tag{2.44}$$

into (2.42a). We find that the complex amplitude is

$$A_{\rm P} = \frac{F}{(\sigma^2 - \omega_f^2) + i\,\varepsilon\omega_f}\ , \qquad \equiv F\,\Lambda e^{i\chi}\ , \tag{2.45a}$$

so that

$$x_{\rm P} = \Lambda\left[F e^{i(\omega_f t + \chi)} + F^* e^{-i(\omega_f t + \chi)}\right]\ , \tag{2.45b}$$

where

$$\Lambda^2 = \frac{1}{(\sigma^2 - \omega_f^2)^2 + \varepsilon^2\omega_f^2}\ ,$$

$$\sin\chi = \frac{-\,\varepsilon\omega_f}{\sqrt{(\sigma^2 - \omega_f^2)^2 + \varepsilon^2\omega_f^2}}\ , \tag{2.46a,b,c}$$

$$\cos\chi = \frac{\sigma^2 - \omega_f^2}{\sqrt{(\sigma^2 - \omega_f^2)^2 + \varepsilon^2\omega_f^2}}\ .$$

Therefore dissipation produces a phase lag, χ, between the forcing and the response. We show below that this lag is essential in enabling the motion to extract energy from the forcing so that the oscillation is sustained against the damping. Note that because $\sin\chi < 0$ the angle χ is in the interval $-\pi < \chi < 0$.

The homogeneous solution for (2.43) is the same as in (2.10):

$$x_{\rm H} = e^{-\varepsilon t/2}\,[A_{\rm H} e^{i\sigma_\varepsilon t} + A_{\rm H}^* e^{-i\sigma_\varepsilon t}]\ , \qquad \sigma_\varepsilon \equiv \sqrt{\sigma^2 - \varepsilon^2/4}\ , \tag{2.47}$$

where $A_{\rm H}$ and $A_{\rm H}^*$ are determined so that the initial conditions are satisfied. The most important point to note is that the homogeneous solution decays exponentially after the initial imposition of the forcing so that at large times $x \approx x_{\rm P}$ no matter what the initial conditions happen to be. One says that the homogeneous solution is *transient*.

If the dissipation is not too large, (2.45) exhibits a *resonance* when the frequency of the forcing, ω_f, is close to the frequency of the free mode, σ. Figure II.4 shows the energy (proportional to Λ^2) of the response as a function of the forcing frequency ω_f. It is easy to show that the width of the peak in Figure II.4, taken at the half-energy points, is approximately ε. The important result to remember is that if the damping is weak (i.e., small ε and high Q, where previously $Q \sim \sigma/\varepsilon$) and the forcing frequency equals the natural frequency then the forced vibration is very large, and the width of the peak is very narrow - the

Figure II.4. Energy of SHO response versus forcing frequency. [To be added.]
Go to the following web site to experiment with an interactive applet simulating a forced, damped harmonic oscillator: http://www.walter-fendt.de/ph14e/resonance.htm .

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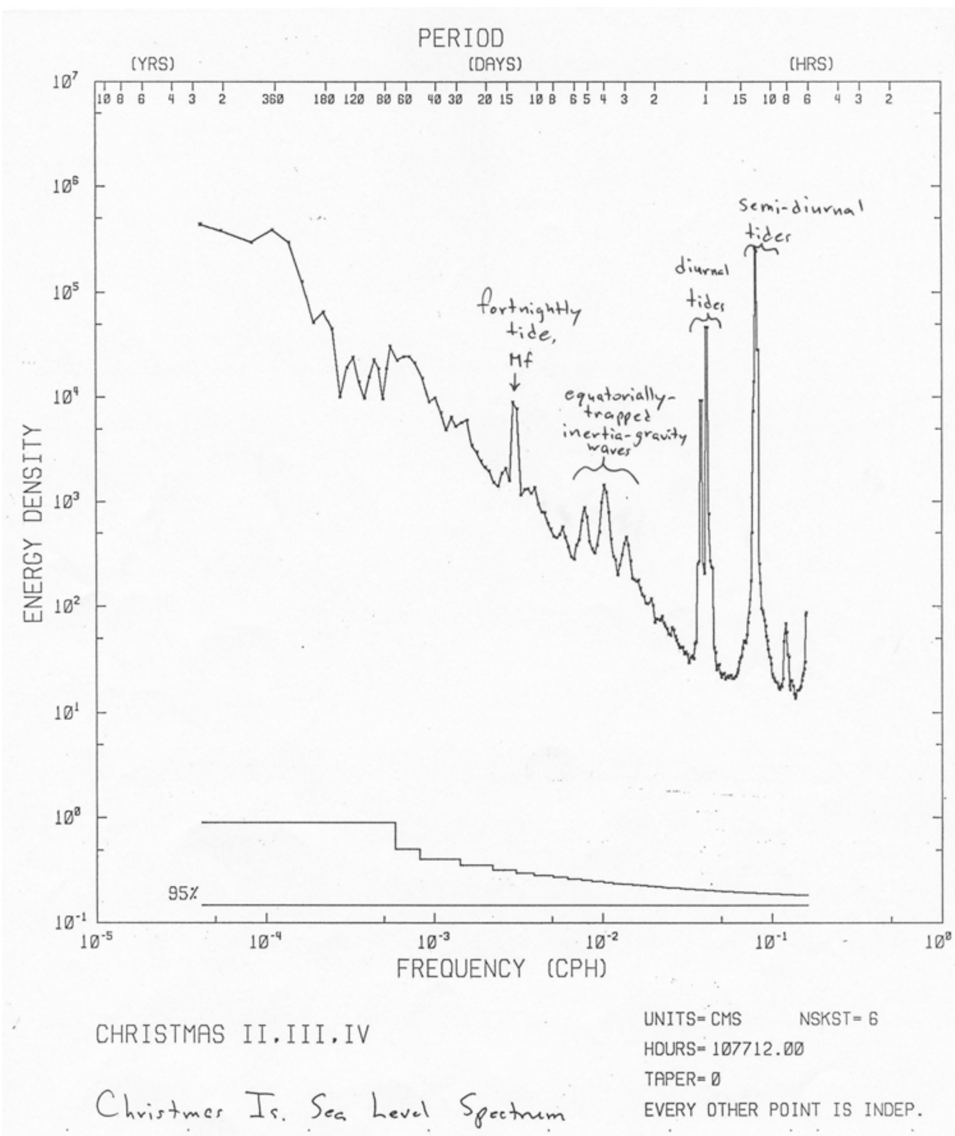
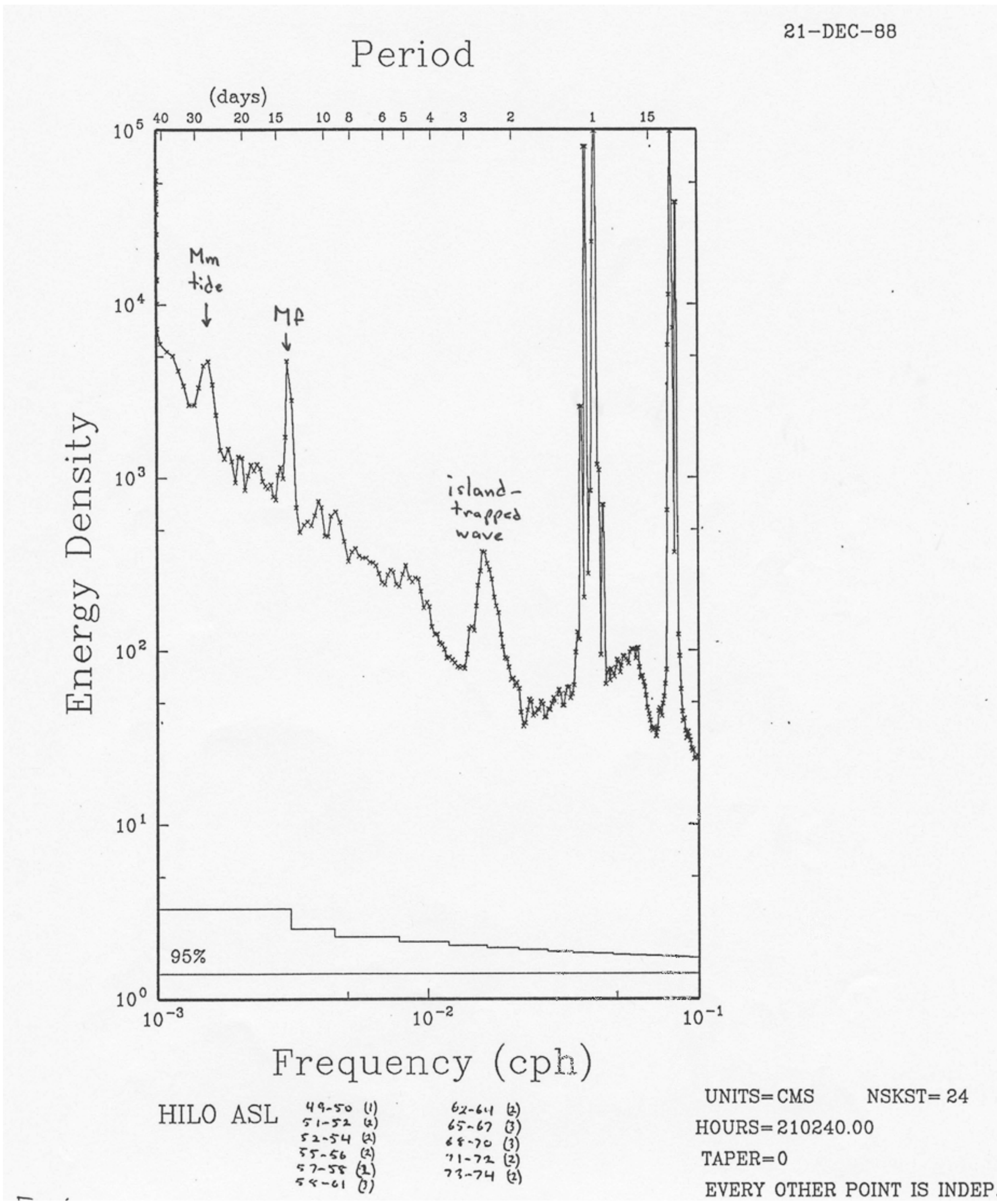
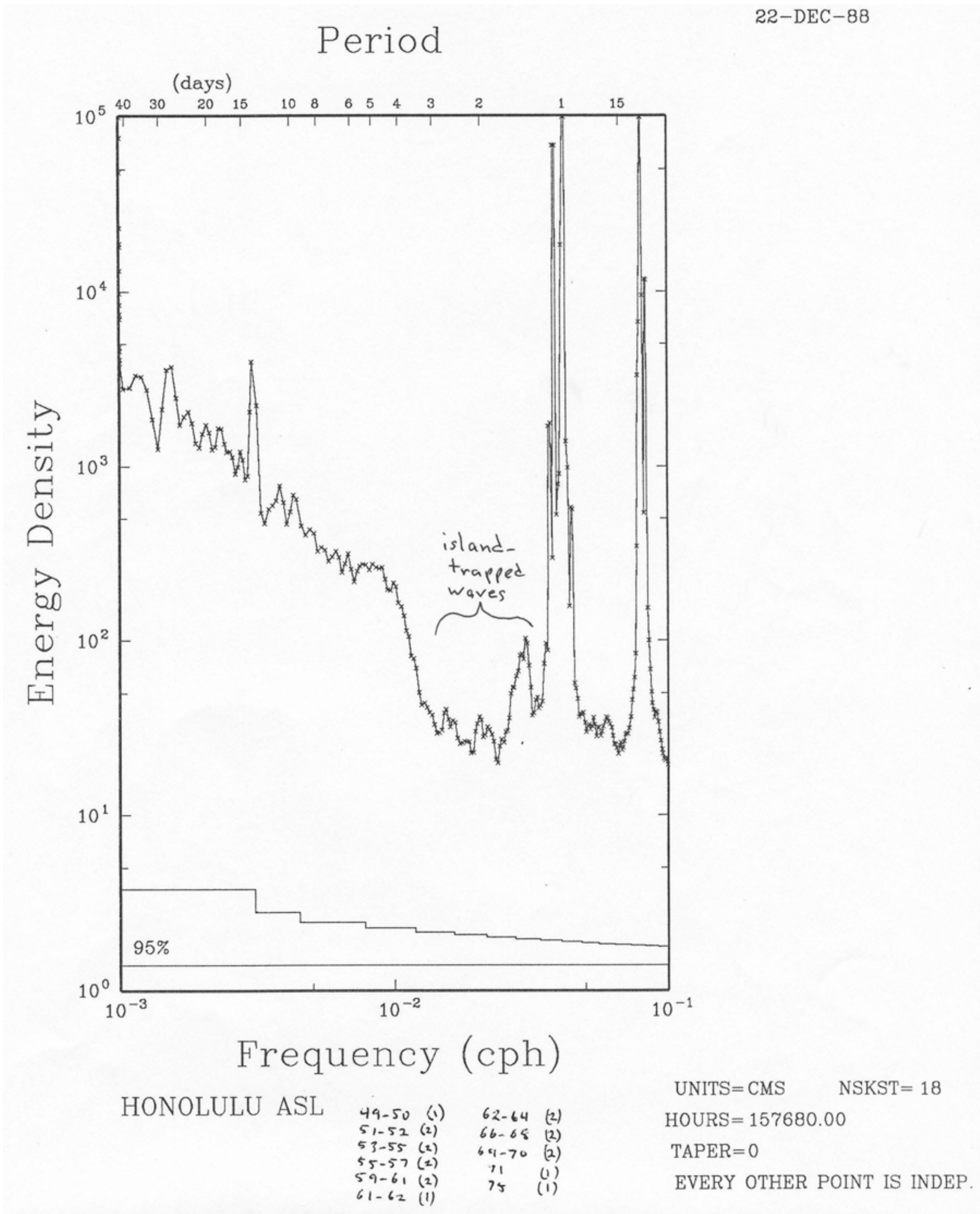
Extra Figure 3 - Sea Level power spectral density from Christmas Is. (now Kiritimati; 01°52′N 157°24′W) in the Pacific Ocean. The prominent oscillations in sea level are indicated. In addition to the tides, the spectrum exhibits narrow-band spectral peaks at the expected pseudo-resonant periods (3-5 days) of equatorially-trapped internal waves. The resonant quality, Q, of these oscillations is estimated to be 8-19 (under-damped). The spectrum was estimated from ~12 years of hourly sea level data. The 95% confidence limits are the distances between the lines at the bottom and should be applied to independent points. Every other point plotted is independent.



Extra Figure 4 - Sea Level power spectral density from Hilo, Hawaii. The prominent oscillations in sea level are indicated. In addition to the tides, the spectrum exhibits narrow-band spectral peaks at the expected pseudo-resonant periods (17 and 59 hrs) of coastally-trapped internal waves (Kelvin Waves) for the island of Hawaii. The resonant quality, Q, of these oscillations is estimated to be 4-7. The spectrum was estimated from ~24 years of hourly sea level data. The 95% confidence limits are the distances between the lines at the bottom and should be applied to independent points. Every other point plotted is independent.



Extra Figure 5 - Sea Level power spectral density from Honolulu, Hawaii. In addition to the tides, the spectrum exhibits narrow-band spectral peaks at the expected pseudo-resonant periods (17, 35 and 47 hrs) of coastally-trapped internal waves (Kelvin Waves) for Oahu Is. The resonant quality, Q, of these oscillations is estimated to be 4-6. The spectrum was estimated from ~18 years of hourly sea level data. The 95% confidence limits are the distances between the lines at the bottom and should be applied to independent points. Every other point plotted is independent.



resonance is sharp.

D.2 The Energy Balance in Forced Oscillations

Because the particular solution in (2.44)-(2.46) is the ultimate motion that is established after the transient decays it is important to understand its energy balance. The time average of (2.7) over a period, $T = 2\pi/\omega_f$, is

$$\epsilon\overline{\dot{x}^2} = \overline{\dot{x}f} \quad , \tag{2.48}$$

which shows that forcing balances dissipation (overbar denotes time average). Given our exact solution we can compute the terms in (2.48). Using (2.44)-(2.46), we have

$$\overline{\dot{x}f} = -2\Lambda FF^* \omega_f \sin\chi \ . \tag{2.49}$$

First, we see that $\sin\chi$ has to be negative (as anticipated), or equivalently $-\pi \le \chi \le 0$, so that the energy input is positive. We have already remarked on this in our discussion of (2.46b) but now we see that the sign of χ has an important physical significance. Because $-\pi \le \chi \le 0$ the maximum response (i.e., maximum displacement) occurs some time less than half a period *after* the maximum forcing (using (2.44)-(2.46), convince yourself of this by creating a real solution from a real forcing). Second, from (2.46), we see that close to resonance $\chi \approx -\pi/2$ so that the forcing and the displacement, x, are a quarter cycle out of phase, while the forcing and the velocity, $\dot{x}$, are in phase (the forcing is always acting in the direction of motion). This means that when the $\sin$ in (2.49) is very close to -1 there is an efficient extraction of energy from the forcing. As a result the response is very large unless the dissipation is strong.

D.3 Periodic Forcing Functions and Resonance

Now that we can solve the oscillator equation with the sinusoidal forcing function in (2.42b), we can use linear superposition and Fourier series to solve the equation for the large and important class of *periodic* forcing functions. An example is the "square wave forcing"

$$f = F\,\mathrm{sqw}(\alpha t) \ , \tag{2.50}$$

where the function $\mathrm{sqw}(x)$ is defined in Fig. II.5. The first thing we do is simplify the notation by introducing non-dimensional variables

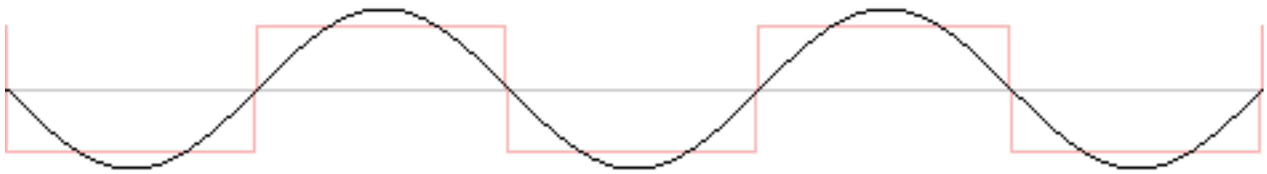
$$\hat{t} = \alpha t \ , \quad \hat{x} = \alpha^2 F^{-1} x \ , \quad \hat{\epsilon} = \epsilon/\alpha \ , \quad \hat{\sigma} = \sigma/\alpha \ . \tag{2.51}$$

The non-dimensional form of (2.42a) with (2.50) is then

$$\ddot{x} + \epsilon\dot{x} + \sigma^2 x = \mathrm{sqw}(t) \ , \tag{2.52}$$

where we have now dropped all the ^'s. (They can be restored later if necessary).

1 harmonic



3 harmonics (1+3+5)

6 harmonics (1+3+5+7+9+11)

15 harmonics (1+3+5+7+9+11+13+15+17+19+21+23+25+27+29)

Figure II.5. Square wave function representation as a sum of sinusoids.
Go to the following web site to see a graphic demonstration of the reconstruction of a square wave with sinusoids: http://en.wikipedia.org/wiki/File:SquareWave.gif .

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Figure II.6. Dissipation of forced, damped harmonic oscillator versus frequency.
[To be added.]

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Next we expand the forcing function, sqw(t), in a Fourier series

$$\mathrm{sqw}(t) = \frac{1}{2}\sum_{n=-\infty}^{n=\infty} a_n e^{int} \ , \qquad a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-imt}\,\mathrm{sqw}(t)\,\mathrm{d}t \ . \qquad (2.53a,b)$$

The integral in (2.53b) is readily evaluated, yielding

$$\mathrm{sqw}(t) = -\frac{2i}{\pi}\sum_{m\ \mathrm{odd}} \frac{1}{m} e^{imt} = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right] . (2.54)$$

Finally, the solution of (2.52) is simply a linear superposition of solutions like (2.45), viz.,

$$x(t) = -\frac{2i}{\pi}\sum_{m\ \mathrm{odd}} \frac{e^{imt}}{m[\sigma^2 - m^2 + i\varepsilon m]} \ , \ \dot{x}(t) = \frac{2}{\pi}\sum_{m\ \mathrm{odd}} \frac{e^{imt}}{\sigma^2 - m^2 + i\varepsilon m} . \ (2.55)$$

Notice that the coefficients in the Fourier series for $\mathrm{sqw}(t)$ decay rather slowly - only as m^{-1} - while the coefficients in the Fourier series for $x(t)$ in (2.55) decay much more quickly - as m^{-3}. If you try to sum these series you'll soon appreciate that as a result of this difference the series in (2.55) converge much more rapidly than the series in (2.54). The function $\mathrm{sqw}(t)$ is discontinuous and its Fourier coefficients decay slowly because it is difficult for a sum of smooth, infinitely-differentiable sinusoids like $\exp(imt)$ to equal a discontinuous function. Figure II.5 illustrates the convergence of the sum in (2.54) to $\mathrm{sqw}(t)$. The solution of (2.52) is continuous and, in fact, so is the first derivative $\dot{x}$ - it is only $\ddot{x}$ that is discontinuous! Thus $x(t)$ is "smoother" than $\mathrm{sqw}(t)$ and consequently its Fourier series is more rapidly convergent.

To calculate the rate at which the square wave forcing does work, as per (2.48), we compute average dissipation over a period:

$$\begin{aligned}
\varepsilon\overline{\dot{x}^2} &\equiv \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} \dot{x}^2\,\mathrm{d}t \ , \\
&= \frac{2\varepsilon}{\pi^3}\int_{-\pi}^{\pi} \sum_{m\ \mathrm{odd}} \frac{e^{imt}}{(-m^2 + i\varepsilon m + \sigma^2)} \sum_{n\ \mathrm{odd}} \frac{e^{int}}{(-n^2 + i\varepsilon n + \sigma^2)}\,\mathrm{d}t \ , \quad (2.56) \\
&= \frac{4\varepsilon}{\pi^2}\sum_{m\ \mathrm{odd}} \frac{1}{(\sigma^2 - m^2)^2 + \varepsilon^2 m^2} \ .
\end{aligned}$$

The dissipation has been graphed as a function of σ in Figure II.6. Note the "resonance peaks" at $\sigma = 1, 3, 5$, etc. The Fourier series in (2.54) shows that the square wave forcing function is a sum of sinusoids with periods $T_1 = 2\pi$, $T_3 = 2\pi/3$, $T_5 = 2\pi/5$, etc. If the natural period of the oscillator, $T = 2\pi/\sigma$, matches the period of *any* of these components in the forcing then there is a very large response.

E. The Green's Function of the Oscillator

E.1 Impulse Response and Linear Superposition

Up to this point we have solved (2.6) with a limited number of forcing functions. There are several different and complementary approaches to finding a general representation of the solution with an arbitrary $f(t)$. In this section we explain one of these: the Green's function approach. The following material is easier to understand if you have assimilated *Exercises 1.3 and 1.4*.

The basic idea of the Green's function is to represent $f(t)$ as a sum of "building blocks" as shown in Figure II.7. In this figure the time axis is divided into intervals of width T centered on the points t_i. In each of these intervals $f(t)$ is approximated by $f(t_i)$. Provided that T is small (and eventually we take the limit $T \to 0$) this approximation is good. Formally, we express $f(t)$ as

$$f(t) \approx \sum_{i=-\infty}^{\infty} f(t_i)\chi_i(t) \ , \tag{2.57}$$

where $\chi_i(t)$ is the *characteristic function* of the interval $t_i - T/2 < t < t_i + T/2$:

$$\chi_i(t) = 1 \quad \text{if} \quad t_i - \frac{T}{2} < t < t_i + \frac{T}{2} \quad \text{and} \quad \chi_i(t) = 0 \quad \text{otherwise.} \tag{2.58}$$

From *Exercise 1.4* we know how to solve

$$\ddot{x}_i + \varepsilon\dot{x}_i + \sigma^2 x_i = \chi_i(t) \ . \tag{2.59}$$

If $T \ll \sigma^{-1}, \varepsilon^{-1}$ then the solution of (2.59) is

$$x_i(t) = T\,G(t - t_i) \ , \tag{2.60}$$

where

$$G(t) = \sigma_\varepsilon^{-1}\sin(\sigma_\varepsilon t)e^{-\varepsilon t/2}H(t) \ . \tag{2.61}$$

In the limit $T \to 0$ the right hand side of (2.59) is called an *impulsive forcing function* and the response in (2.61) is the *impulse response* or *Green's function*.

Using linear superposition the solution with the forcing function in (2.57) is

$$x(t) = \sum_{i=-\infty}^{\infty} f(t_i)T\,G(t - t_i) \ . \tag{2.62}$$

The final step is to take the limit $T \to 0$ so that (2.62) becomes an integral ($t_i \to t_1$,

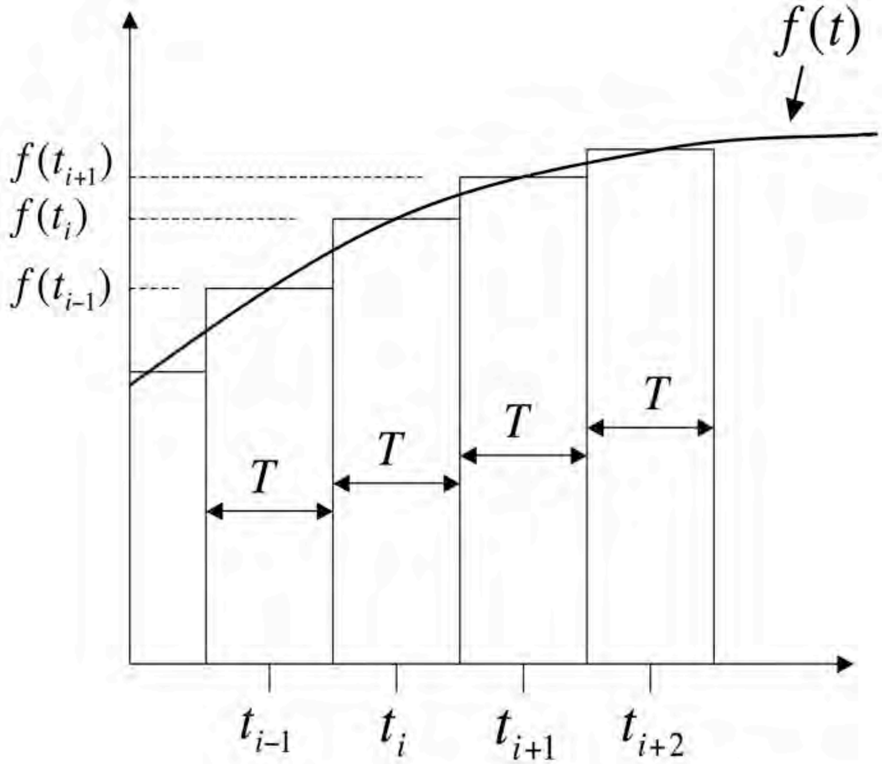

Figure II.7. A representation of the function *f(t)* in terms of a sum of building blocks with appropriate amplitudes, per equation 2.57. As $T \rightarrow 0$, the accuracy of the representation improves.

$T \to \mathrm{d}t_1, f(t_i) \to f(t_1)$). The result is

$$x(t) = \int_{-\infty}^{\infty} f(t_1) G(t - t_1) \mathrm{d}t_1 \ . \tag{2.63}$$

Equation (2.63) is an *integral representation* of the solution of (2.6). You should check this by direct substitution. (In your previous courses on differential equations you may have seen (2.63) obtained using *variation of parameters*.)

E.2 Construction of the δ-function as the Limit of a Sequence of Functions

The route we took to (2.63) was well described in the 19th century. A more modern approach uses the concept of a "δ-function". A δ-function is not really a function, but rather a *sequence* of functions such as

$$\delta_1(t,T) \equiv \frac{1}{T\sqrt{\pi}} e^{-t^2/T^2} \qquad \text{or} \qquad \delta_2(t,T) = T^{-1}\chi_0(t) \ . \tag{2.64a,b}$$

In (2.64b), $\chi_0(t)$ is the characteristic function of the interval $-T/2 < t < T/2$. In both of the examples in (2.64) there is a parameter T that selects a different member of the sequence of functions. A δ-function sequence has the property that

$$\int_{-\infty}^{\infty} \delta(t,T) \mathrm{d}t = 1 \ , \tag{2.65}$$

i.e., the integral of each member of the sequence is independent of the parameter T. (The integral in (2.65) is constant because the width of the peak is proportional to T while its height is proportional to T^{-1} so that the area, which is the product of width and height, is independent of T.)

The second property of a δ-sequence is called the *sifting property*. If $f(t)$ is some arbitrary function then

$$\int_{-\infty}^{\infty} f(t_1)\delta(t - t_1)\mathrm{d}t_1 \equiv \lim_{T \to 0} \int_{-\infty}^{\infty} f(t_1)\delta(t - t_1, T)\mathrm{d}t_1 = f(t) \ . \tag{2.66}$$

This works because as $T \to 0$ the functions in (2.64) become increasingly concentrated around $t = 0$ as illustrated in Figure II.8.

The "δ-function", $\delta(t)$, means the limit of a δ-sequence when $T \to 0$. For instance, to "solve the equation"

$$\ddot{G} + \varepsilon\dot{G} + \sigma^2 G = \delta(t) \tag{2.67}$$

really means to solve the *sequence* of equations

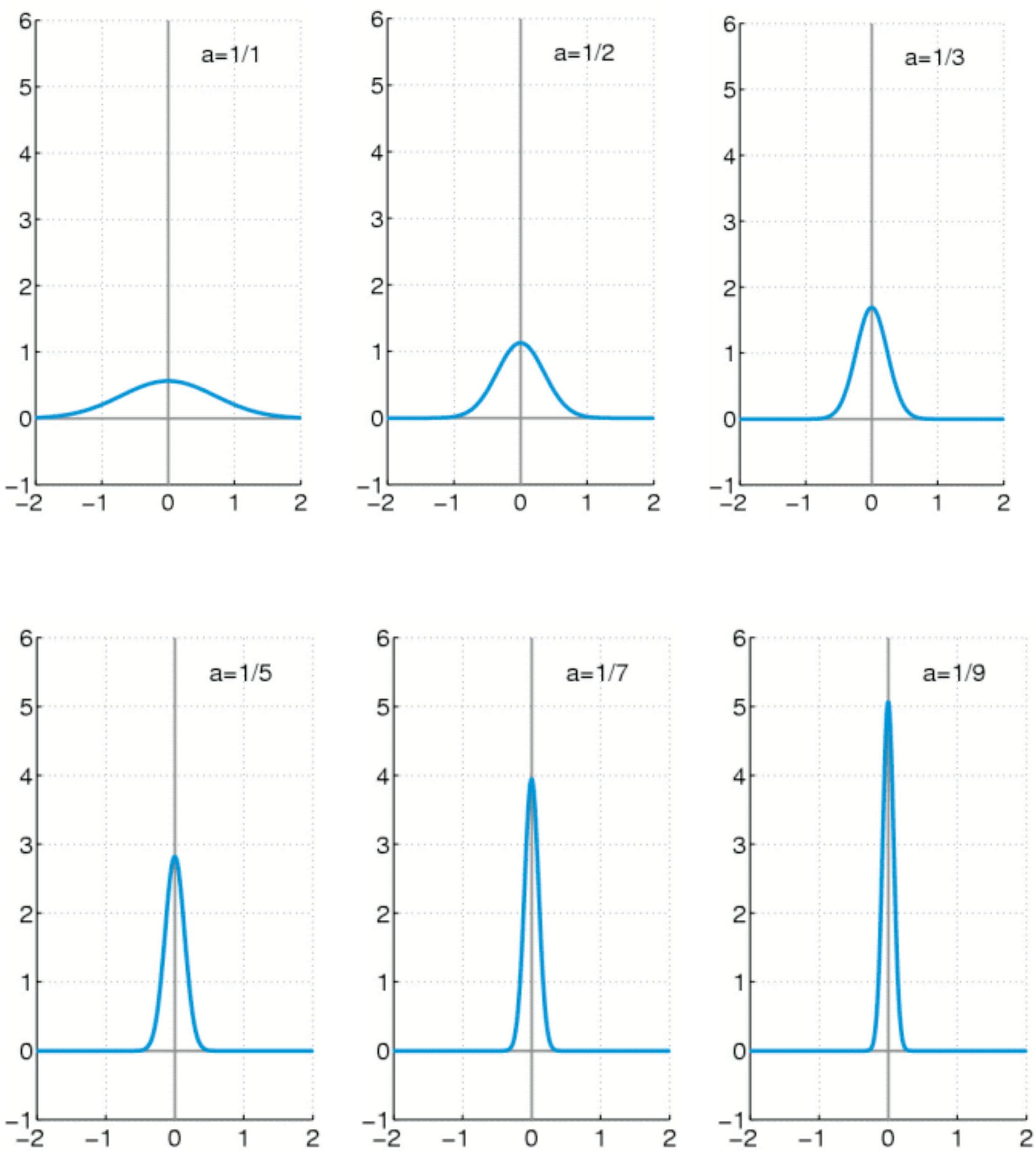

Figure II.8. Dirac delta function sequence: $\delta(t,a) = \dfrac{1}{a\sqrt{\pi}} e^{-t^2/a^2}$

Go to the following web site to see a graphic demonstration of this delta function sequence: http://en.wikipedia.org/wiki/Dirac_delta_function

$$\ddot{G}(t,T) + \varepsilon\dot{G}(t,T) + \sigma^2 G(t,T) = \delta(t,T)\ , \tag{2.68}$$

and then take the limit $T \to 0$. The solution of (2.67) is

$$G(t) = \lim_{T \to 0} G(t,T) = \sigma_\varepsilon^{-1}\sin(\sigma_\varepsilon t)e^{-\varepsilon t/2}H(t)\ . \tag{2.69}$$

If $G(t)$ is the solution of (2.67) it is also now obvious that $x(t)$ in (2.63) is the solution of (2.6). Check it by substitution; you'll need to use the sifting property of δ-functions in (2.66).

Here is a list of some of the more useful properties of the δ-function:

$$f(t)\delta(t-t_1) = f(t_1)\delta(t-t_1),\quad t\delta(t) = 0,\quad \delta(-t) = \delta(t), \tag{2.70a}$$

$$\delta(at) = |a|^{-1}\delta(t),\quad \delta(t^2-a^2) = \frac{1}{2|a|}[\delta(t-a) + \delta(t+a)]\ . \tag{2.70b}$$

δ-functions have the inverse dimension of their argument, i.e., $\delta(t)$ has the dimension $(\text{time})^{-1}$. Integration by parts shows that the derivative of the δ-function has the property

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\mathrm{d}t = -f'(0)\ . \tag{2.71}$$

You should also think about the following:

$$\frac{\mathrm{d}(H(t)t)}{\mathrm{d}t} = H(t)\ ,\qquad \frac{\mathrm{d}H(t)}{\mathrm{d}t} = \delta(t)\ . \tag{2.72a,b}$$

Are equations (2.72a,b) consistent with the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}[tH(t)] = \frac{\mathrm{d}t}{\mathrm{d}t}H(t) + t\frac{\mathrm{d}H(t)}{\mathrm{d}t}\ ? \tag{2.73}$$

E.3 Jump Conditions

Recall how we calculated (2.61): we solved (2.68) and then took $T \to 0$. Fortunately, it is not necessary to do this every time you see a δ-function - there are certain techniques that enable you to go directly to the limit of the sequence without explicitly computing every member of it, and then taking the limit. But conceptually a δ-function is defined by a limiting process and every equation containing it can be interpreted in this way. This interpretation is often an aid to our physical visualization of δ-functions as sudden impulsive forces. The amazing thing is that δ-functions and Green's functions can be used as building blocks to construct *any* smooth, slowly varying function: this is the content of equations (2.57), (2.62) and their continuous analogs (2.63) and (2.66).

To solve (2.67) directly we first note that the forcing term on the right hand side is zero except in the neighborhood of $t = 0$. Thus the solution when $t < 0$ is simply $G(t) = 0$ -

there is no forcing to excite the oscillator. Then at $t=0$ the oscillator gets an impulsive kick from the δ-function. The effect of the kick is calculated by integrating (2.67) over a small time interval that surrounds $t=0$, say $-\tau<t<\tau$. Because $\dot{G}(-\tau)=G(-\tau)=0$ the result of this integration is

$$\dot{G}(\tau) + \varepsilon G(\tau) + \sigma^2 \int_{-\tau}^{\tau} G(t)\mathrm{d}t = 1 \ . \tag{2.74}$$

Now as $\tau \to 0$ one of the three terms on the left hand side must balance the one on the right hand side. Some thought shows that it must be $\dot{G}(\tau)$ since the other two terms vanish as $\tau \to 0$. This means there is a jump in $\dot{G}$ at $t=0$: $G(t)$ is continuous, $\dot{G}(t)$ is discontinuous at $t=0$ and $\ddot{G}(t)$ has a δ-function component at $t=0$. (As (2.72b) shows, the derivative of a discontinuous function is a δ-function.) [Once again note the general principle that higher derivatives of a function become increasingly singular: in an equation like (2.67) it is always the most highly differentiated term (in this case $\ddot{G}$) that balances the δ-function. The δ-function cannot be balanced by $\dot{G}$ since then $\ddot{G}$ would be a δ-function derivative and there is nothing left in the equation to balance this horrible singularity.]

To summarize this argument: the effect of δ-function forcing on the right hand side of (2.67) is to produce a jump in $\dot{G}(t)$ at $t=0$:

$$G(0^-) = 0 \ , \quad G(0^+) = 0 \ , \quad \dot{G}(0^-) = 0 \ , \quad \dot{G}(0^+) = 1 \ . \tag{2.75}$$

The conditions at $t=0^+$ can then be used as "effective initial conditions" for the homogeneous equation $\ddot{G} + \varepsilon\dot{G} + \sigma^2 G = 0$ and the solution of this problem is given by (2.61). Thus if you understand how to derive *jump conditions* such as (2.75) then there is no need to go through all of the intermediate steps of solving the differential equation for a δ-sequence and taking limits, etc.